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Conway's Normal Form in the Mizar System

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Summary. This paper presents a formal definition of the Conway normal form, a structured representation uniquely suited to characterising surreal numbers by expressing them as sums within a hierarchically ordered group. To this end, we formalise the first sections of the chapter *The Structure of the General Surreal Number* in Conway's book. In particular, we define omega maps and prove the existence and uniqueness of the Conway name for surreal numbers.

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Introduction

Conway surreal numbers can be constructed according to two independent principles: the game-theoretic approach ([5, 9]) and the tree-theoretic approach ([6, 7]). In this formalization we use our construction of the $\approx$ equivalence class representative of a surreal number x, called *uniq-surreal*, denoted as $\mathrm{Uniq}_{\mathbf{No}}x$ ([11]), to unify these two approaches in order to formalize the canonical representation, called normal forms by Conway. The definition of the *Conway Normal Form* allows an analysis of the structure of surreal numbers as an ordered vector space over $\mathbb{R}$. This framework provides a path for future research on surreal numbers, as it allows e.g. the reformulation of basic arithmetic operations in terms of vector space operations, thus facilitating the application of vector space theory to the analysis of surreal numbers.

The formalization follows [5, 8, 7], selected fragments have been described in [13].

In Section 1, we define the relation between two numbers, x and y, as commensurate if and only if $x < n \cdot y$ and $y < m \cdot x$, for some $n, m \in \mathbb{N}^+$ (see Def.1). Then we prove that this relation is both an equivalence and a convex relation. Conway defines this relation using only one natural number, which is equivalent to our approach (see Th7). Additionally, we define and prove the fundamental property of the *infinitesimal less* operator (see Def.2), as follows: $x <^\infty y$ if $x \cdot n < y$ for all $n \in \mathbb{N}^+$.

Section 2 introduces the Conway $\boldsymbol{\omega}$-map ([5]) and demonstrates the fundamental property $\boldsymbol{\omega}^0 = 1$ (see Th26), $\boldsymbol{\omega}^{(x+y)} \approx \boldsymbol{\omega}^x \cdot \boldsymbol{\omega}^y$ (see Th27) which are typical for the standard power function. Note that it has an additional properties, namely that $\boldsymbol{\omega}^x <^\infty \boldsymbol{\omega}^y$ for $x < y$. We also examines the behaviour of $\boldsymbol{\omega}$-map in the context of the commensurate and infinitesimal less relations, as well as applying the standard absolute value to extend context for negative surreal numbers.

In Section 3 we prove the existence of the unique characterization of non zero surreal numbers x as pairs consisting of a commensurate leader y and a non zero real number r for a given $x \not\approx 0$ such that $|x - r \cdot \boldsymbol{\omega}^y| <^\infty |x|$. We define $\boldsymbol{\omega}_r(x) = r$, $\boldsymbol{\omega}_y(x) = y$ (see Def.7, Def.8)

In the following section, we direct our attention to the convex subclass of surreal numbers differing from s by infinitesimal less than $\boldsymbol{\omega}^y$, which is referred to as the β-term in Conway's handbook, where β is defined in the context of this work. We say that x is $(s, y, r)-$term if and only if $|x - (s + r \cdot \boldsymbol{\omega}^y)| <^\infty \boldsymbol{\omega}^y$ (Def.12) Note that our definition of the β-term is based on an explanation provided by Ehrlich ([8], Theorem 13).

In Section 5, in accordance with Ehrlich's approach, we formally introduce the convex subclass $\bigcap \boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r}, \alpha$ as follows:

$$x \text{ is } \bigcap \boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r}, \alpha \iff \forall_{\beta<\alpha}\, x \text{ is } (\boldsymbol{s}(\beta), \boldsymbol{y}(\beta), \boldsymbol{r}(\beta)) - \text{term},$$

where $\boldsymbol{s}$, $\boldsymbol{y}$ are sequences of surreal numbers and $\boldsymbol{r}$ is real numbers, each of at least α-length. Next proceed to assume that the length of $\boldsymbol{s}$ is at least $\alpha + 1$. A triple $(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r})$ *simplest on* α if $\alpha = 0$ and $\boldsymbol{s}(\alpha) = 0$ or $\alpha \neq 0$, $\boldsymbol{s}(\alpha)$ is $\bigcap \boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r}, \alpha$ and has the smallest birth of all $\bigcap \boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r}, \alpha$ surreal numbers (see Def.15). Additionally, we call a triple $(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r})$ *simplest up to* α if $(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r}, \beta)$ simplest for all $\beta < \alpha$ (see Def.16). This section is concluded with the proof of two properties of the sequence $\boldsymbol{s}$. Firstly, we demonstrate that the sequence $\boldsymbol{s}$ is unique up to position α if it contains only uniq-surreal numbers and if $(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r})$ is in its simplest up to α (Th77,Th80) Secondly, we provide an example of a sequence $\boldsymbol{s}$ of uniq-surreal numbers for which $(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r})$ is simplest up to α under the assumption that $\boldsymbol{y}$ is a strictly decreasing sequence, and that $\boldsymbol{r}$ is a sequence of non-zero real numbers (Th82). Using these properties, we define Conway's generalistion of partial sums as follows.

Definition 1 (Def.18) *Let α be an ordinal, $\boldsymbol{y} = \{\boldsymbol{y}_\beta\}_{\beta<\alpha}$ be a strictly decreasing sequence of surreal numbers, $\boldsymbol{r} = \{\boldsymbol{r}_\beta\}_{\beta<\alpha}$ be a sequence of non-zero real. Consider $\boldsymbol{s} = \{\boldsymbol{s}_\beta\}_{\beta\leqslant\alpha}$ where the triple $(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{r})$ is simplest up to α. For each each $\beta \leqslant \alpha$ we define unique expression $\sum_{\gamma<\beta} \boldsymbol{\omega}^{\boldsymbol{y}_\gamma} \cdot \boldsymbol{r}_\gamma$ to be $\boldsymbol{s}_\beta$ called βth Conway's partial sum.*

In Section 6, we concentrate on the approximation of a given number $x \not\approx 0$ using commensurate leaders. Applying $\boldsymbol{\omega}$-maps, we get $x_1 \coloneqq x - \boldsymbol{\omega}^{\boldsymbol{y}_0} \cdot \boldsymbol{r}_0$ which is infinitely smaller in absolute terms than x where $\boldsymbol{r}_0 \coloneqq \boldsymbol{\omega}_r(x)$, $\boldsymbol{y}_0 \coloneqq \boldsymbol{\omega}_y(x)$. Then, if $x_1 \not\approx 0$, it is possible to produce another $\boldsymbol{r}_1$, $\boldsymbol{y}_1$, x_2 in a similar manner where $|x_2| <^\infty |x_1| <^\infty |x|$ and $x = \boldsymbol{\omega}^{\boldsymbol{y}_0} \cdot \boldsymbol{r}_0 + \boldsymbol{\omega}^{\boldsymbol{y}_1} \cdot \boldsymbol{r}_1 + x_2$. We call the constructed sequences $(\boldsymbol{r}, \boldsymbol{y})$ the α-*name* of x if the remainder is non-zero in each iteration β for $\beta < \alpha$ where α is an ordinal. As we illustrated in Theorem Th101, for any a strictly decreasing sequence of surreal numbers $\boldsymbol{y} = \{\boldsymbol{y}_\beta\}_{\beta<\alpha}$ and a sequence of non-zero real $\boldsymbol{r} = \{\boldsymbol{r}_\beta\}_{\beta<\alpha}$, $(\boldsymbol{r}, \boldsymbol{y})$ is the α-name of $\sum_{\beta<\alpha} \boldsymbol{\omega}^{\boldsymbol{y}_\beta} \cdot \boldsymbol{r}_\beta$. We constructed also an ordinal α and two α-length sequences $(\boldsymbol{r}, \boldsymbol{y})$, for a given x such that $\sum_{\beta<\alpha} \boldsymbol{\omega}^{\boldsymbol{y}_\beta} \cdot \boldsymbol{r}_\beta \approx x$ (see Th100). Finally, we prove that this pair of sequences is unique (see Th102), known as the Conway Normal Form ([5]).

1. Commensurability in Archimedean Classes of Surreal Numbers

From now on A, B denote ordinal numbers, o denotes an object, x, y, z denote surreal numbers, n denotes a natural number, and r, r_1, r_2 denote real numbers.

Now we state the proposition:

(1) (uReal)$(r) \in$ Dayω.

The functor No-omega yielding a No-ordinal unique surreal number is defined by the term

(Def. 1) No-uOrdinal-op(ω).

Let x, y be surreal numbers. We say that x, y are commensurate if and only if

(Def. 2) there exists a positive natural number n such that $x <$ (uInt)$(n) \cdot y$ and there exists a positive natural number n such that $y <$ (uInt)$(n) \cdot x$.

Observe that the predicate is symmetric.

Now we state the propositions:

(2) If x is positive, then x, x are commensurate.

(3) If x, y are commensurate, then x is positive.

Let us consider surreal numbers x, y, z. Now we state the propositions:

(4) If x, y are commensurate and y, z are commensurate, then x, z are commensurate.
PROOF: There exists a positive natural number n such that $x < (\mathrm{uInt})(n) \cdot z$ by [12, (70), (51), (69)], [**?** , (15)]. Consider n being a positive natural number such that $y < (\mathrm{uInt})(n) \cdot x$. Consider m being a positive natural number such that $z < (\mathrm{uInt})(m) \cdot y$. □

(5) If $x \approx y$ and x, z are commensurate, then y, z are commensurate.
PROOF: There exists a positive natural number n such that $y < (\mathrm{uInt})(n) \cdot z$ by [11, (4)]. Consider n being a positive natural number such that $z < (\mathrm{uInt})(n) \cdot x$. □

(6) If x, z are commensurate and $x \leqslant y \leqslant z$, then x, y are commensurate and y, z are commensurate. The theorem is a consequence of (3), (5), and (2).

Now we state the propositions:

(7) x, y are commensurate if and only if there exists a positive natural number n such that $x < (\mathrm{uInt})(n) \cdot y$ and $y < (\mathrm{uInt})(n) \cdot x$.
PROOF: If x, y are commensurate, then there exists a positive natural number n such that $x < (\mathrm{uInt})(n) \cdot y$ and $y < (\mathrm{uInt})(n) \cdot x$ by (3), [12, (70)], [**?** , (9)], [11, (4)]. □

(8) If x is positive and $x \approx y$, then x, y are commensurate.

Let x, y be surreal numbers. We say that x infinitely$\langle$ y if and only if

(Def. 3) for every positive real number r, $x \cdot (\mathrm{uReal})(r) < y$.

Now we state the propositions:

(9) If x infinitely$\langle$ y, then $x < y$.

(10) Let us consider a real number r. Then $(\mathrm{uReal})(r)$ infinitely$\langle$ No-omega. The theorem is a consequence of (1).

Let us consider surreal numbers x, y, z. Now we state the propositions:

(11) If $x \leqslant y$ infinitely$\langle$ z, then x infinitely$\langle$ z.

(12) If x infinitely$\langle$ $y \leqslant z$, then x infinitely$\langle$ z.

Now we state the propositions:

(13) Let us consider a positive real number r, and surreal numbers x, y. Suppose x infinitely$\langle$ y. Then
  (i) $x \cdot (\mathrm{uReal})(r)$ infinitely$\langle$ y, and
  (ii) x infinitely$\langle$ $y \cdot (\mathrm{uReal})(r)$.

PROOF: $x \cdot (\mathrm{uReal})(r)$ infinitely$\langle$ y by [**?** , (57)], [12, (69), (51)], [11, (4)]. □

(14) Let us consider surreal numbers x, y, z. Suppose $x$infinitely$\langle y$infinitely$\langle z$. Then x infinitely$\langle$ z.

(15) If x, y are commensurate and y infinitely$\langle$ z, then x infinitely$\langle$ z.

(16) If x, y are commensurate and z infinitely$\langle$ x, then z infinitely$\langle$ y.

(17) If $x \approx y$ and y infinitely$\langle$ z, then x infinitely$\langle$ z.

(18) If x infinitely$\langle$ z and y infinitely$\langle$ z, then $x + y$ infinitely$\langle$ z. The theorem is a consequence of (13).

(19) If $x \approx y$ and z infinitely$\langle$ x, then z infinitely$\langle$ y.

(20) If $\mathbf{0_{No}} \leqslant x$ infinitely$\langle$ y, then $x \cdot (\text{uReal})(r) < y$. The theorem is a consequence of (9).

2. Conway's ω-map

Let us consider A. The functor $\boxed{\text{No-omega-op}(A)}$ yielding a many sorted set indexed by Day$A$ is defined by

(Def. 4) there exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\operatorname{dom} S = \operatorname{succ} A$ and $it = S(A)$ and for every B such that $B \in \operatorname{succ} A$ there exists a many sorted set S_1 indexed by Day$B$ such that $S(B) = S_1$ and for every object x such that $x \in \text{Day}B$ holds $S_1(x) = \langle\{\mathbf{0_{No}}\} \cup \{(\bigcup \operatorname{rng}(S{\restriction}B))(x_3){*}(\text{uReal})(r)$, where x_3 is an element of L_x, r is an element of $\mathbb{R} : x_3 \in \mathrm{L}_x$ and r is positive$\}$, $\{(\bigcup \operatorname{rng}(S{\restriction}B))(x_4) * (\text{uReal})(r)$, where x_4 is an element of R_x, r is an element of $\mathbb{R} : x_4 \in \mathrm{R}_x$ and r is positive$\}\rangle$.

Now we state the proposition:

(21) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence S. Suppose for every B such that $B \in \operatorname{dom} S$ there exists a many sorted set S_1 indexed by Day$B$ such that $S(B) = S_1$ and for every object x such that $x \in \text{Day}B$ holds $S_1(x) = \langle\{\mathbf{0_{No}}\} \cup \{(\bigcup \operatorname{rng}(S{\restriction}B))(x_3) * (\text{uReal})(r)$, where x_3 is an element of L_x, r is an element of $\mathbb{R} : x_3 \in \mathrm{L}_x$ and r is positive$\}$, $\{(\bigcup \operatorname{rng}(S{\restriction}B))(x_4){*}(\text{uReal})(r)$, where x_4 is an element of R_x, r is an element of $\mathbb{R} : x_4 \in \mathrm{R}_x$ and r is positive$\}\rangle$. If $A \in \operatorname{dom} S$, then No-omega-op$(A) = S(A)$.

Proof: Define $\mathcal{D}$(ordinal number) $=$ Day$\$_1$. Define $\mathcal{H}$(object, $\subseteq$-monotone , function yielding transfinite sequence) $= \langle\{\mathbf{0_{No}}\} \cup \{(\bigcup \operatorname{rng} \$_2)(x_3){*}(\text{uReal})(r)$, where x_3 is an element of $\mathrm{L}_{\$_1}, r$ is an element of $\mathbb{R} : x_3 \in \mathrm{L}_{\$_1}$ and r is positive$\}$, $\{(\bigcup \operatorname{rng} \$_2)(x_4){*}(\text{uReal})(r)$, where x_4 is an element of $\mathrm{R}_{\$_1}, r$ is an element of $\mathbb{R} : x_4 \in \mathrm{R}_{\$_1}$ and r is positive$\}\rangle$. Consider S_2 being a $\subseteq$-monotone, function yielding transfinite sequence such that $\operatorname{dom} S_2 = \operatorname{succ} A$ and $S_2(A) = $ No-omega-op(A) and for every ordinal number B such that

$B \in \operatorname{succ} A$ there exists a many sorted set S_1 indexed by $\mathcal{D}(B)$ such that $S_2(B) = S_1$ and for every object x such that $x \in \mathcal{D}(B)$ holds $S_1(x) = \mathcal{H}(x, S_2 \restriction B)$. $S1 \restriction \operatorname{succ} A = S_2 \restriction \operatorname{succ} A$ from [12, Sch. 2]. $\square$

Let us consider x. The functor No-omegaˆ(x) yielding a set is defined by the term

(Def. 5) $(\text{No-omega-op}(\mathfrak{born}\, x))(x)$.

Let us observe that No-omegaˆ(x) is surreal and No-omegaˆ(x) is positive.

Now we state the propositions:

(22) $o \in \mathrm{L}_{\text{No-omega}^\wedge(\text{x})}$ if and only if $o = \mathbf{0}_{\mathbf{No}}$ or there exists a surreal number x_3 and there exists a positive real number r such that $x_3 \in \mathrm{L}_x$ and $o = (\text{No-omega}^\wedge(\text{x}_3)) \cdot (\text{uReal})(\text{r})$.

(23) $o \in \mathrm{R}_{\text{No-omega}^\wedge(\text{x})}$ if and only if there exists a surreal number x_4 and there exists a positive real number r such that $x_4 \in \mathrm{R}_x$ and $o = (\text{No-omega}^\wedge(\text{x}_4)) \cdot (\text{uReal})(\text{r}$

(24) If $x \leqslant y$, then $\text{No-omega}^\wedge(\text{x}) \leqslant \text{No-omega}^\wedge(\text{y})$.

(25) If $x < y$, then $\text{No-omega}^\wedge(\text{x})$ infinitely⟨ $\text{No-omega}^\wedge(\text{y})$.

(26) $\text{No-omega}^\wedge(\mathbf{0}_{\mathbf{No}}) = \mathbf{1}_{\mathbf{No}}$. The theorem is a consequence of (22) and (23).

(27) $(\text{No-omega}^\wedge(\text{x})) \cdot (\text{No-omega}^\wedge(\text{y})) \approx \text{No-omega}^\wedge(\text{x} + \text{y})$.

Proof: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every surreal numbers x, y such that $\mathfrak{born}\, x \oplus \mathfrak{born}\, y = \$_1$ holds $(\text{No-omega}^\wedge(\text{x})) \cdot (\text{No-omega}^\wedge(\text{y})) \approx \text{No-omega}^\wedge(\text{x} + \text{y}$ For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$ by [12, (50), (28)], (23), [**?** , (55), (48)]. For every ordinal number D, $\mathcal{P}[D]$ from [2, Sch. 2]. $\square$

(28) $(\text{No-omega}^\wedge(\text{x}))^{-1} \approx \text{No-omega}^\wedge(-\text{x})$. The theorem is a consequence of (26) and (27).

(29) Let us consider surreal numbers x_3, x. Suppose $x_3 \leqslant x$ and x_3, $\text{No-omega}^\wedge(\text{y})$ are commensurate and x, $\text{No-omega}^\wedge(\text{y})$ are not commensurate. Then $\text{No-omega}^\wedge(\text{y})$ infinitely⟨ x.

(30) Let us consider surreal numbers x, x_4. Suppose $\mathbf{0}_{\mathbf{No}} < x \leqslant x_4$ and x_4, $\text{No-omega}^\wedge(\text{y})$ are commensurate and x, $\text{No-omega}^\wedge(\text{y})$ are not commensurate. Then x infinitely⟨ $\text{No-omega}^\wedge(\text{y})$.

Let x be a surreal number. The functor $|x|$ yielding a surreal number is defined by the term

(Def. 6) $\begin{cases} x, & \textbf{if } \mathbf{0}_{\mathbf{No}} \leqslant x, \\ -x, & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(31) $\mathbf{0}_{\mathbf{No}} \leqslant |x|$.

(32) (i) $|x| = x$, or

(ii) $|x| = -x$.

(33) $x \approx \mathbf{0_{No}}$ if and only if $|x| \approx \mathbf{0_{No}}$.

(34) $-|x| \leqslant x \leqslant |x|$.

(35) $-y \leqslant x \leqslant y$ if and only if $|x| \leqslant y$.

Proof: If $-y \leqslant x \leqslant y$, then $|x| \leqslant y$ by [12, (10)]. $\mathbf{0_{No}} \leqslant |x|$. $\square$

(36) If $x \not\approx \mathbf{0_{No}}$, then $|x|$ is positive.

(37) $|x + y| \leqslant |x| + |y|$. The theorem is a consequence of (34) and (35).

(38) If $x \approx \mathbf{0_{No}}$, then $|-x| \approx |x|$.

(39) If $x \not\approx \mathbf{0_{No}}$, then $|-x| = |x|$.

(40) $|-x| \approx |x|$.

(41) If $|x|$infinitely$\langle z$ and $|y|$infinitely$\langle z$, then $|x+y|$infinitely$\langle z$. The theorem is a consequence of (13) and (37).

(42) If $|x|$ infinitely$\langle$ z, then $|-x|$ infinitely$\langle$ z. The theorem is a consequence of (40).

(43) If $|x|$infinitely$\langle z$ and $|y|$infinitely$\langle z$, then $|x-y|$infinitely$\langle z$. The theorem is a consequence of (42) and (41).

Now we state the propositions:

(44) If $|y|$ infinitely$\langle$ x, then $x + y \not\approx \mathbf{0_{No}}$. The theorem is a consequence of (9).

(45) If $|y|$ infinitely$\langle$ $|x|$, then $x + y \not\approx \mathbf{0_{No}}$. The theorem is a consequence of (44), (40), and (17).

(46) If $|y|$ infinitely$\langle$ x, then $x \not\approx \mathbf{0_{No}}$. The theorem is a consequence of (9) and (31).

(47) If $|y|$ infinitely$\langle$ $|x|$, then $x \not\approx \mathbf{0_{No}}$. The theorem is a consequence of (46).

Now we state the propositions:

(48) If $x \approx y$, then $|x| \approx |y|$.

(49) $||x| - |y|| \leqslant |x - y|$. The theorem is a consequence of (37), (48), (39), (38), and (35).

(50) $||x|| = |x|$.

(51) If $x \leqslant y \leqslant z$, then $|y| \leqslant |x| + |z|$. The theorem is a consequence of (31).

(52) $-y < x < y$ if and only if $|x| < y$.

Proof: If $-y < x < y$, then $|x| < y$ by [12, (10)]. $\mathbf{0_{No}} \leqslant |x|$. $\square$

(53) If $\mathbf{0_{No}} \leqslant x$ infinitely$\langle$ y, then $|x \cdot (\mathrm{uReal})(r)|$ infinitely$\langle$ y. The theorem is a consequence of (20).

3. Unique Characterization of Surreal Number

Let x be a surreal number. Assume $x \not\approx \mathbf{0}_{\mathbf{No}}$. The functor omega-y($x$) yielding an unique surreal number is defined by

(Def. 7) $|x|$, No-omegaˆ(it) are commensurate.

Now we state the propositions:

(54) Suppose x, No-omegaˆ(y) are commensurate. Then there exists a positive real number s such that $|x -$ (No-omegaˆ(y)) $\cdot$ (uReal)(s)$|$ infinitely$\langle$ x.
PROOF: Set $N =$ No-omegaˆ(y). Define $\mathcal{L}$[object] $\equiv \$_1$ is a real number and for every real number r such that $r = \$_1$ holds $N \cdot$ (uReal)(r) $\leqslant x$. Define $\mathcal{R}$[object] $\equiv \$_1$ is a real number and for every real number r such that $r = \$_1$ holds $x < N \cdot$ (uReal)(r). For every extended reals r, s such that $\mathcal{L}[r]$ and $\mathcal{R}[s]$ holds $r \leqslant s$ by [11, (4)], [**?** , (51)], [12, (75)]. Consider s being an extended real such that for every extended real r such that $\mathcal{L}[r]$ holds $r \leqslant s$ and for every extended real r such that $\mathcal{R}[r]$ holds $s \leqslant r$. Consider n being a positive natural number such that $x <$ (uInt)(n) $\cdot N$ and $N <$ (uInt)(n) $\cdot x$. $\square$

(55) If x is positive and $|x -$ (No-omegaˆ(y)) $\cdot$ (uReal)(r)$|$ infinitely$\langle$ x, then r is positive. The theorem is a consequence of (9).

(56) If $x \not\approx \mathbf{0}_{\mathbf{No}}$, then omega-y($x$) $=$ omega-y($-x$). The theorem is a consequence of (39).

Let x be a surreal number. Assume $x \not\approx \mathbf{0}_{\mathbf{No}}$. The functor omega-r($x$) yielding a non zero real number is defined by

(Def. 8) $|x -$ (No-omegaˆ(omega-y(x))) $\cdot$ (uReal)(it)$|$ infinitely$\langle$ $|$x$|$.

Now we state the propositions:

(57) Let us consider a positive natural number n. Suppose $|y| \cdot$(uReal)($\frac{n+1}{n}$) $<$ $|x|$. Then $|x|$, $|x+y|$ are commensurate. The theorem is a consequence of (31), (39), (38), (49), and (37).

(58) If $|x|$ is positive, then $x \not\approx \mathbf{0}_{\mathbf{No}}$.

(59) If $x \cdot$ (uReal)(r_1) $< y \cdot$ (uReal)(r_2) and $0 < r$, then $x \cdot$ (uReal)($r_1 \cdot r$) $<$ $y \cdot$ (uReal)($r_2 \cdot r$).

(60) If $x \cdot$ (uReal)(r_1) $\leqslant y \cdot$ (uReal)(r_2) and $0 \leqslant r$, then $x \cdot$ (uReal)($r_1 \cdot r$) $\leqslant$ $y \cdot$ (uReal)($r_2 \cdot r$).

(61) Suppose $x \not\approx \mathbf{0}_{\mathbf{No}}$ and $y \not\approx \mathbf{0}_{\mathbf{No}}$. Then omega-y($x$) $=$ omega-y(y) if and only if $|x|$, $|y|$ are commensurate. The theorem is a consequence of (4).

(62) Suppose $x \not\approx \mathbf{0}_{\mathbf{No}}$ and $x + y \not\approx \mathbf{0}_{\mathbf{No}}$ and omega-y(x) $=$ omega-y($x+y$) and omega-r(x) $=$ omega-r($x+y$). Then $|y|$ infinitely$\langle$ $|x|$. The theorem is a consequence of (16), (4), (48), (37), (59), and (40).

(63) Suppose $|y|$ infinitely$\langle$ $|x|$. Then

(i) $x \not\approx \mathbf{0_{No}}$, and

(ii) $x + y \not\approx \mathbf{0_{No}}$, and

(iii) omega-y(x) = omega-y$(x + y)$, and

(iv) omega-r(x) = omega-r$(x + y)$.

PROOF: $|x|$, $|x + y|$ are commensurate. Set N = No-omega^(omega-y(x)). $|x + y + -N \cdot$ (uReal)(omega-r$(x))|$ infinitely$\langle$ $|x|$ by [12, (37)], (37), [12, (67), (75)]. $|x + y - N \cdot$ (uReal)(omega-r$(x))|$ infinitely$\langle$ $|x + y|$. $\square$

(64) If $x \not\approx \mathbf{0_{No}}$ and $y \approx \mathbf{0_{No}}$, then y infinitely$\langle$ $|x|$. The theorem is a consequence of (36).

(65) If (uReal)$(r) \approx \mathbf{0_{No}}$, then $r = 0$.

(66) If x is positive and $r \neq 0$, then $|$(uReal)$(r) \cdot x|$, x are commensurate. The theorem is a consequence of (48), (39), (38), and (5).

The scheme *Simplest* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists an unique surreal number s such that $\mathcal{P}[s]$ and for every unique surreal number x such that $\mathcal{P}[x]$ and $x \neq s$ holds $\mathfrak{b}$orn $s \in \mathfrak{b}$orn x

provided

- there exists a surreal number x such that $\mathcal{P}[x]$ and
- for every surreal numbers x, y, z such that $x \leqslant y \leqslant z$ and $\mathcal{P}[x]$ and $\mathcal{P}[z]$ holds $\mathcal{P}[y]$.

Let f be a function. We say that f is surreal-valued if and only if

(Def. 9) rng f is surreal-membered.

Let s be a surreal number. Note that $\langle s \rangle$ is surreal-valued and there exists a transfinite sequence which is surreal-valued.

Let f be a surreal-valued function. Let us note that rng f is surreal-membered.

A Surreal-Sequence is a surreal-valued transfinite sequence. Let X be a surreal-membered set. Observe that every subset of X is surreal-membered.

Let f be a surreal-valued function and X be a set. One can check that $f{\restriction}X$ is surreal-valued.

Let f, g be Surreal-Sequences. One can verify that $f \frown g$ is surreal-valued.

Let f be a function. We say that f is uniq-surreal-valued if and only if

(Def. 10) rng f is unique surreal-membered.

Let s be an unique surreal number. One can check that $\langle s \rangle$ is uniq-surreal-valued and there exists a transfinite sequence which is uniq-surreal-valued.

Let f be an uniq-surreal-valued function. Note that rng f is unique surreal-membered.

An uSurreal-Sequence is an uniq-surreal-valued transfinite sequence. Let X be an unique surreal-membered set. Let us note that every subset of X is unique surreal-membered.

Let f be an uniq-surreal-valued function and X be a set. One can verify that $f{\restriction}X$ is uniq-surreal-valued.

Let f, g be uSurreal-Sequences. Let us observe that $f \frown g$ is uniq-surreal-valued and every set which is unique surreal-membered is also surreal-membered and every function which is uniq-surreal-valued is also surreal-valued.

Let S be a Surreal-Sequence. We say that S is strictly decreasing if and only if

(Def. 11) for every ordinal numbers a, b such that $a \in b \in \operatorname{dom} S$ for every surreal numbers s_5, s_6 such that $s_5 = S(a)$ and $s_6 = S(b)$ holds $s_6 < s_5$.

Let s be an unique surreal number. Let us note that $\langle s \rangle$ is strictly decreasing and there exists an uSurreal-Sequence which is strictly decreasing.

Let s_1, s_2 be non-zero transfinite sequences. Let us note that $s_1 \frown s_2$ is non-zero and there exists a transfinite sequence of elements of $\mathbb{R}$ which is non-zero.

4. α-term - An Essential Component of the Conway Normal Form

Let s be an object, y be a surreal number, r be a real number, and x be an object. We say that x is (s,y,r)-term if and only if

(Def. 12) $x +' -'s \not\approx \mathbf{0_{No}}$ and omega-y$(x +' -'s) \approx y$ and omega-r$(x +' -'s) = r$.

Let s, y be surreal numbers and x be a surreal number. One can check that x is (s,y,r)-term if and only if the condition (Def. 13) is satisfied.

(Def. 13) $x - s \not\approx \mathbf{0_{No}}$ and omega-y$(x - s) \approx y$ and omega-r$(x - s) = r$.

Now we state the propositions:

(67) If $r \neq 0$, then (uReal)$(r) \cdot$ (No-omega^(y)) $\not\approx \mathbf{0_{No}}$. The theorem is a consequence of (66) and (3).

(68) If $r \neq 0$, then omega-y((uReal)$(r) \cdot$(No-omega^(y))) $=$ Unique$_{\mathbf{No}}$(y). The theorem is a consequence of (66), (67), and (5).

(69) Let us consider a surreal number s. Suppose $r \neq 0$. Then $s +$ (uReal)$(r) \cdot$ (No-omega^(y)) is (s,y,r)-term. The theorem is a consequence of (67), (68), (36), (48), (8), (61), and (64).

(70) Suppose $x \approx y$ and $x \not\approx \mathbf{0_{No}}$. Then

(i) omega-y$(x) =$ omega-y(y), and

(ii) omega-r$(x) =$ omega-r(y).

The theorem is a consequence of (36), (48), (8), (61), (16), and (17).

Let us consider a surreal number s. Now we state the propositions:

(71) Suppose $r \neq 0$. Then $s + (\text{uReal})(r) \cdot (\text{No-omega\^{}(y)}) + \text{x}$ is (s,y,r)-term if and only if $|x|$ infinitely$\langle$ No-omega^(y).

Proof: Set $N =$ No-omega^(y). Set $R = (\text{uReal})(r)$. Set $s_{10} = s+R\cdot N+x$. Set $s_9 = s + R \cdot N + -s$. $R \cdot N \not\approx \mathbf{0_{No}}$. $|s_9|$ is positive. $|s_9|$, $|N \cdot R|$ are commensurate. $|N\cdot R|$, N are commensurate. $|s_9|$, N are commensurate. $s+R\cdot N$ is (s,y,r)-term. If s_{10} is (s,y,r)-term, then $|x|$infinitely$\langle N$ by [11, (4)], (70), [11, (50)], (16). $|x|$ infinitely$\langle$ $|s_9|$. $s_9 + x \not\approx \mathbf{0_{No}}$ and omega-y(s_9) = omega-y($s_9 + x$) and omega-r(s_9) = omega-r($s_9 + x$). □

(72) If $r \neq 0$ and x is (s,y,r)-term and $x \approx z$, then z is (s,y,r)-term. The theorem is a consequence of (70).

(73) Suppose $r \neq 0$. Then x is (s,y,r)-term if and only if $|x - (s + (\text{uReal})(r) \cdot (\text{No-omega\^{}(y)}))|$ infinitely$\langle$ No-omega^(y). The theorem is a consequence of (72) and (71).

Now we state the proposition:

(74) Let us consider surreal numbers s, p. Suppose $r \neq 0$. Let us consider surreal numbers x, y, z. Suppose x is (s,p,r)-term and z is (s,p,r)-term and $x \leqslant y \leqslant z$. Then y is (s,p,r)-term. The theorem is a consequence of (73), (18), (11), and (51).

5. Conway's Generalization of Partial Sums

Let r be a transfinite sequence of elements of $\mathbb{R}$, y, s be transfinite sequences, α be an ordinal number, and x be a surreal number. We say that x in meets terms s, y, r, α if and only if

(Def. 14) for every ordinal number β and for every surreal numbers s_6, y_4 such that $\beta \in \alpha$ and $s_6 = s(\beta)$ and $y_4 = y(\beta)$ holds x is (s_6,y_4,($r(\beta)$))-term.

We say that s, y, r simplest on position α if and only if

(Def. 15) for every surreal number s_5 such that $s_5 = s(\alpha)$ holds if $0 = \alpha$, then $s_5 = \mathbf{0_{No}}$ and if $0 \neq \alpha$, then s_5 in meets terms s, y, r, α and for every unique surreal number x such that x in meets terms s, y, r, α and $x \neq s_5$ holds $\mathfrak{born}\, s_5 \in \mathfrak{born}\, x$.

Let us consider a transfinite sequence r of elements of $\mathbb{R}$, transfinite sequences y, s_1, s_2, and an ordinal number α. Now we state the propositions:

(75) If $s_1{\restriction}\alpha = s_2{\restriction}\alpha$ and x in meets terms s_1, y, r, α, then x in meets terms s_2, y, r, α.

(76) Suppose $s_1(\alpha)$ is an unique surreal number and $s_2(\alpha)$ is an unique surreal number and $s_1{\restriction}\alpha = s_2{\restriction}\alpha$ and s_1, y, r simplest on position α and s_2, y, r

simplest on position α. Then $s_1(\alpha) = s_2(\alpha)$. The theorem is a consequence of (75).

Let r be a transfinite sequence of elements of $\mathbb{R}$, y, s be transfinite sequences, and α be an ordinal number. We say that s, y, r simplest up to α if and only if

(Def. 16) for every ordinal number β such that $\beta \in \alpha$ holds s, y, r simplest on position β.

Now we state the propositions:

(77) Let us consider a transfinite sequence r of elements of $\mathbb{R}$, a transfinite sequence y, uSurreal-Sequences s_1, s_2, and an ordinal number α. Suppose $\alpha \subseteq \operatorname{dom} s_1$ and $\alpha \subseteq \operatorname{dom} s_2$ and s_1, y, r simplest up to α and s_2, y, r simplest up to α. Then $s_1 \restriction \alpha = s_2 \restriction \alpha$.

Proof: Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \in \alpha$, then $s_1(\$_1) = s_2(\$_1)$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$ by [14, (62)], [4, (49), (2)], (76). For every ordinal number D, $\mathcal{P}[D]$ from [2, Sch. 2]. For every object x such that $x \in \alpha$ holds $(s_1 \restriction \alpha)(x) = (s_2 \restriction \alpha)(x)$ by [4, (49)]. $\square$

(78) Let us consider a transfinite sequence r of elements of $\mathbb{R}$, transfinite sequences y, s, and ordinal numbers α, β. Suppose $\beta \subseteq \alpha$ and s, y, r simplest up to α. Then s, y, r simplest up to β.

Let us consider a transfinite sequence r of elements of $\mathbb{R}$, transfinite sequences y, s, and an ordinal number α. Now we state the propositions:

(79) x in meets terms s, y, r, α if and only if x in meets terms $s \restriction \operatorname{succ} \alpha$, y, r, α.

Proof: If x in meets terms s, y, r, α, then x in meets terms $s \restriction \operatorname{succ} \alpha$, y, r, α by [2, (8)], [4, (49)]. $\square$

(80) $s \restriction \operatorname{succ} \alpha$, y, r simplest on position α if and only if s, y, r simplest on position α. The theorem is a consequence of (79).

Now we state the propositions:

(81) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, transfinite sequences p, s, and an ordinal number α. Suppose $\alpha \subseteq \operatorname{dom} r$. Let us consider surreal numbers x, y, z. Suppose $x \leqslant y \leqslant z$ and x in meets terms s, p, r, α and z in meets terms s, p, r, α. Then y in meets terms s, p, r, α. The theorem is a consequence of (74).

(82) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, and a strictly decreasing Surreal-Sequence y. Then there exists an uSurreal-Sequence s such that

(i) $\operatorname{dom} s = \operatorname{succ}(\operatorname{dom} r \cap \operatorname{dom} y)$, and

(ii) s, y, r simplest up to $\operatorname{dom} s$.

Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \subseteq \operatorname{dom} r \cap \operatorname{dom} y$, then there exists an uSurreal-Sequence s such that $\operatorname{dom} s = \operatorname{succ} \$_1$ and s, y, r simplest up to $\operatorname{dom} s$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$ by [2, (8), (6), (28)], [14, (62)]. For every ordinal number D, $\mathcal{P}[D]$ from [2, Sch. 2]. $\square$

Let r be a non-zero transfinite sequence of elements of $\mathbb{R}$ and y be a strictly decreasing Surreal-Sequence. The functor Partial-Sums(r, y) yielding an uSurreal-Sequence is defined by

(Def. 17) $\operatorname{dom} it = \operatorname{succ}(\operatorname{dom} r \cap \operatorname{dom} y)$ and for every A such that $A \in \operatorname{dom} it$ holds it, y, r simplest on position A.

The functor $\sum_{\kappa=0}^{y} r(\kappa)$ yielding an unique surreal number is defined by the term

(Def. 18) $(\text{Partial-Sums}(r, y))(\operatorname{dom} r \cap \operatorname{dom} y)$.

Let s be a strictly decreasing Surreal-Sequence and A be an ordinal number. Note that $s \restriction A$ is strictly decreasing.

Let R be a non-zero binary relation and X be a set. Let us note that $R \restriction X$ is non-zero.

Let us consider a transfinite sequence r of elements of $\mathbb{R}$, transfinite sequences y, s, and ordinal numbers A, B. Now we state the propositions:

(83) If $A \subseteq B$, then x in meets terms s, y, r, A iff x in meets terms s, $y \restriction B$, $r \restriction B$, A.

Proof: If x in meets terms s, y, r, α, then x in meets terms s, $y \restriction B$, $r \restriction B$, α by [4, (49)]. $\square$

(84) If $B \subseteq A$, then s, $y \restriction A$, $r \restriction A$ simplest on position B iff s, y, r simplest on position B. The theorem is a consequence of (83).

Now we state the proposition:

(85) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing Surreal-Sequence y, and an ordinal number A. Then $\text{Partial-Sums}(r, y) \restriction \operatorname{succ} A = \text{Partial-Sums}(r \restriction A, y \restriction A)$.

Proof: $\operatorname{succ}(\operatorname{dom} r \cap \operatorname{dom} y) \cap \operatorname{succ} A = \operatorname{succ}(\operatorname{dom}(r \restriction A) \cap \operatorname{dom}(y \restriction A))$ by [2, (16), (22), (21)]. $\text{Partial-Sums}(r, y) \restriction \operatorname{succ} A$, $y \restriction A$, $r \restriction A$ simplest up to $\operatorname{dom}(\text{Partial-Sums}(r, y) \restriction \operatorname{succ} A)$ by [2, (21), (22)], [14, (74)], (84). $\square$

6. Conway Names for Surreal Numbers

Let r be a non-zero transfinite sequence of elements of $\mathbb{R}$, y be a strictly decreasing Surreal-Sequence, α be an ordinal number, and x be a surreal number.

We say that r, y, α name like x if and only if

(Def. 19) $\alpha \subseteq \operatorname{dom} r = \operatorname{dom} y$ and for every ordinal number β such that $\beta \in \alpha$ for every surreal number P_1 such that $P_1 = (\text{Partial-Sums}(r, y))(\beta)$ holds $x \not\approx P_1$ and $r(\beta) = \text{omega-r}(x - P_1)$ and $y(\beta) = \text{omega-y}(x - P_1)$.

Now we state the propositions:

(86) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing Surreal-Sequence y, and ordinal numbers α, β. Suppose $\alpha \subseteq \beta$ and r, y, β name like x. Then r, y, α name like x.

(87) Let us consider non-zero transfinite sequences r_1, r_2 of elements of $\mathbb{R}$, strictly decreasing Surreal-Sequences y_1, y_2, and an ordinal number A. Suppose r_1, y_1, A name like x and r_2, y_2, A name like x. Then

(i) $r_1 \restriction A = r_2 \restriction A$, and

(ii) $y_1 \restriction A = y_2 \restriction A$.

Proof: Define $\mathcal{P}[\text{ordinal number}] \equiv$ if r_1, y_1, $\$_1$ name like x and r_2, y_2, $\$_1$ name like x, then $r_1 \restriction \$_1 = r_2 \restriction \$_1$ and $y_1 \restriction \$_1 = y_2 \restriction \$_1$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$ by [14, (62)], [4, (49)], [2, (22), (6)]. For every ordinal number D, $\mathcal{P}[D]$ from [2, Sch. 2]. $\square$

(88) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing Surreal-Sequence y, and an ordinal number A. Suppose r, y, A name like x. Then x in meets terms Partial-Sums(r, y), y, r, A. The theorem is a consequence of (16) and (73).

(89) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, and a strictly decreasing Surreal-Sequence y. Then $\sum_{\kappa=0}^{y} r(\kappa)$ in meets terms Partial-Sums(r, y), y, r, $\operatorname{dom} r \cap \operatorname{dom} y$.

(90) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a transfinite sequence y, a Surreal-Sequence s, and ordinal numbers A, B. Suppose $B \in A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$ and $A \subseteq \operatorname{dom} s$. Let us consider a surreal number y_4. Suppose $y_4 = y(B)$ and x in meets terms s, y, r, A and z in meets terms s, y, r, A. Then $|x - z|$ infinitely$\langle$No-omega^(y_4). The theorem is a consequence of (73), (43), (48), and (11).

(91) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing Surreal-Sequence y, and an ordinal number α. Suppose r, y, α name like x. Then $r \restriction \alpha$, $y \restriction \alpha$, α name like x. The theorem is a consequence of (85).

(92) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, and a strictly decreasing Surreal-Sequence y. Suppose z in meets terms Partial-Sums(r, y), y, r, $\operatorname{dom} r \cap \operatorname{dom} y$ and $z \not\approx \sum_{\kappa=0}^{y} r(\kappa)$. Let us consider

an ordinal number A, and a surreal number y_3. Suppose $A \in \operatorname{dom} r \cap \operatorname{dom} y$ and $y_3 = y(A)$. Then omega-y$(\sum_{\kappa=0}^{y} r(\kappa) - z) < y_3$. The theorem is a consequence of (89), (90), (9), and (15).

(93) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing Surreal-Sequence y, and an ordinal number A. Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then (Partial-Sums$(r, y))(A) = \sum_{\kappa=0}^{y \restriction A} (r \restriction A)(\kappa)$. The theorem is a consequence of (85).

(94) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, and a strictly decreasing Surreal-Sequence y. Suppose x in meets terms Partial-Sums(r, y), y, r, $\operatorname{dom} r \cap \operatorname{dom} y$ and z in meets terms Partial-Sums(r, y), y, r, $\operatorname{dom} r \cap \operatorname{dom} y$ and $x \not\approx z$. Let us consider an ordinal number A, and a surreal number y_3. Suppose $A \in \operatorname{dom} r \cap \operatorname{dom} y$ and $y_3 = y(A)$. Then omega-y$(x - z) < y_3$. The theorem is a consequence of (90), (9), and (15).

(95) Suppose for every non-zero transfinite sequence r of elements of $\mathbb{R}$ and for every strictly decreasing uSurreal-Sequence y such that $\operatorname{dom} r = \operatorname{dom} y$ and r, y, $\operatorname{dom} r$ name like x holds $\sum_{\kappa=0}^{y} r(\kappa) \not\approx x$. Let us consider an ordinal number α. Then there exists a non-zero transfinite sequence r of elements of $\mathbb{R}$ and there exists a strictly decreasing uSurreal-Sequence y such that $\operatorname{dom} r = \operatorname{succ} \alpha = \operatorname{dom} y$ and r, y, $\operatorname{succ} \alpha$ name like x.

PROOF: Define $\mathcal{P}$[ordinal number] $\equiv$ there exists a non-zero transfinite sequence r of elements of $\mathbb{R}$ and there exists a strictly decreasing uSurreal-Sequence y such that $\operatorname{dom} r = \operatorname{succ} \$_1 = \operatorname{dom} y$ and r, y, $\operatorname{succ} \$_1$ name like x. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$ by [2, (6), (21)], [14, (62)], [2, (10)]. For every ordinal number D, $\mathcal{P}[D]$ from [2, Sch. 2]. $\square$

Let s be a Surreal-Sequence. The functor $\mathfrak{b}$orn s yielding a sequence of ordinal numbers is defined by

(Def. 20) $\operatorname{dom} it = \operatorname{dom} s$ and for every ordinal number α such that $\alpha \in \operatorname{dom} s$ for every surreal number s_5 such that $s_5 = s(\alpha)$ holds $it(\alpha) = \mathfrak{b}\text{orn}\, s_5$.

Now we state the proposition:

(96) Let us consider a transfinite sequence r of elements of $\mathbb{R}$, a Surreal-Sequence y, an uSurreal-Sequence s, and an ordinal number A. Suppose s, y, r simplest up to A and $A \subseteq \operatorname{succ} \operatorname{dom} y$. Then $s \restriction A$ is one-to-one.

PROOF: For every ordinal numbers a, b such that $a \in b \in \operatorname{dom}(s \restriction A)$ holds $(s \restriction A)(a) \neq (s \restriction A)(b)$ by [14, (57)], [2, (10)], [4, (47)], [2, (21), (22), (6)]. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(s \restriction A)$ and $(s \restriction A)(x_1) = (s \restriction A)(x_2)$ holds $x_1 = x_2$ by [2, (14)]. $\square$

Let r be a non-zero transfinite sequence of elements of $\mathbb{R}$ and y be a strictly decreasing Surreal-Sequence. One can verify that Partial-Sums(r, y) is one-to-one.

Now we state the proposition:

(97) Let us consider a transfinite sequence r of elements of $\mathbb{R}$, a Surreal-Sequence y, an uSurreal-Sequence s, and an ordinal number α. Suppose s, y, r simplest up to α and $s{\restriction}\alpha$ is one-to-one. Then $\mathfrak{born}\, s{\restriction}\alpha$ is increasing.
PROOF: For every ordinal numbers A, B such that $A \in B \in \mathrm{dom}(\mathfrak{born}\, s{\restriction}\alpha)$ holds $(\mathfrak{born}\, s{\restriction}\alpha)(A) \in (\mathfrak{born}\, s{\restriction}\alpha)(B)$ by [14, (57)], [2, (10)], [4, (49)], [10, (37)]. □

Let r be a non-zero transfinite sequence of elements of $\mathbb{R}$ and y be a strictly decreasing Surreal-Sequence. One can check that $\mathfrak{born}\,\text{Partial-Sums}(r, y)$ is increasing.

Now we state the propositions:

(98) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing Surreal-Sequence y, an uSurreal-Sequence s, and an ordinal number A. Suppose $A \subseteq \mathrm{dom}\, r$ and x in meets terms s, y, r, A and s, y, r simplest up to $\mathrm{succ}\, A$. Then $\mathrm{rng}\, \mathfrak{born}\,(s{\restriction}\mathrm{succ}\, A) \subseteq \mathrm{succ}\, \mathfrak{born}_{\approx} x$. The theorem is a consequence of (81).

(99) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, and a strictly decreasing Surreal-Sequence y. Then $\mathrm{dom}\, r \cap \mathrm{dom}\, y \subseteq \mathfrak{born} \sum_{\kappa=0}^{y} r(\kappa)$.
PROOF: Set $s = \text{Partial-Sums}(r, y)$. $\sum_{\kappa=0}^{y} r(\kappa)$ in meets terms s, y, r, $\mathrm{dom}\, r \cap \mathrm{dom}\, y$ and s, y, r simplest up to $\mathrm{dom}\, s$. $\mathrm{rng}\, \mathfrak{born}\,(s{\restriction}\mathrm{dom}\, s) \subseteq \mathrm{succ}\, \mathfrak{born}_{\approx} \sum_{\kappa=0}^{y} r(\kappa)$. $\mathrm{succ}(\mathrm{dom}\, r \cap \mathrm{dom}\, y) \subseteq \mathrm{succ}\, \mathfrak{born} \sum_{\kappa=0}^{y} r(\kappa)$ by [3, (10)], [2, (12)]. □

Now we state the propositions:

(100) CONWAY NORMAL FORM::
Let us consider a surreal number x. Then there exists a non-zero transfinite sequence r of elements of $\mathbb{R}$ and there exists a strictly decreasing uSurreal-Sequence y such that $\mathrm{dom}\, r = \mathrm{dom}\, y \subseteq \mathfrak{born}_{\approx} x$ and $\sum_{\kappa=0}^{y} r(\kappa) \approx x$.
PROOF: There exists a non-zero transfinite sequence r of elements of $\mathbb{R}$ and there exists a strictly decreasing uSurreal-Sequence y such that $\mathrm{dom}\, r = \mathrm{dom}\, y$ and r, y, $\mathrm{dom}\, r$ name like x and $\sum_{\kappa=0}^{y} r(\kappa) \approx x$ by (95), (88), (98), [1, (11)]. Consider r being a non-zero transfinite sequence of elements of $\mathbb{R}$, y being a strictly decreasing uSurreal-Sequence such that $\mathrm{dom}\, r = \mathrm{dom}\, y$ and r, y, $\mathrm{dom}\, r$ name like x and $\sum_{\kappa=0}^{y} r(\kappa) \approx x$. □

(101) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, and a strictly decreasing uSurreal-Sequence y. Suppose $\mathrm{dom}\, r = \mathrm{dom}\, y$. Then r, y, $\mathrm{dom}\, r$ name like $\sum_{\kappa=0}^{y} r(\kappa)$.
PROOF: Set $s = \sum_{\kappa=0}^{y} r(\kappa)$. $s \not\approx P_1$ by [12, (43), (39)], [11, (4)]. □

(102) Let us consider non-zero transfinite sequences r_1, r_2 of elements of $\mathbb{R}$, and

strictly decreasing uSurreal-Sequences y_1, y_2. Suppose $\operatorname{dom} r_1 = \operatorname{dom} y_1$ and $\operatorname{dom} r_2 = \operatorname{dom} y_2$ and $\sum_{\kappa=0}^{y_1} r_1(\kappa) \approx \sum_{\kappa=0}^{y_2} r_2(\kappa)$. Then

(i) $r_1 = r_2$, and

(ii) $y_1 = y_2$.

The theorem is a consequence of (101), (87), and (85).

(103) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing uSurreal-Sequence y, and an ordinal number A. Suppose $A \subseteq \operatorname{dom} r = \operatorname{dom} y$. Let us consider surreal numbers x, z. If r, y, A name like x and $x \approx z$, then r, y, A name like z. The theorem is a consequence of (70).

Let x be a surreal number. The functor name-ord(x) yielding an ordinal number is defined by

(Def. 21) there exists a non-zero transfinite sequence r of elements of $\mathbb{R}$ and there exists a strictly decreasing uSurreal-Sequence y such that $it = \operatorname{dom} r = \operatorname{dom} y$ and $\sum_{\kappa=0}^{y} r(\kappa) \approx x$.

Now we state the proposition:

(104) Let us consider a non-zero transfinite sequence r of elements of $\mathbb{R}$, a strictly decreasing uSurreal-Sequence y, and a surreal number x. Suppose $\operatorname{dom} r = \operatorname{dom} y$ and $\sum_{\kappa=0}^{y} r(\kappa) \approx x$. Then name-ord($x$) $= \operatorname{dom} r$. The theorem is a consequence of (102).

Let x be a surreal number. The functor name-r(x) yielding a non-zero transfinite sequence of elements of $\mathbb{R}$ is defined by

(Def. 22) there exists a strictly decreasing uSurreal-Sequence y such that $\operatorname{dom} y = \operatorname{dom} it$ and $\sum_{\kappa=0}^{y} it(\kappa) \approx x$.

The functor name-y(x) yielding a strictly decreasing uSurreal-Sequence is defined by

(Def. 23) dom(name-r(x)) $= \operatorname{dom} it$ and $\sum_{\kappa=0}^{it}$ name-r(x)(κ) $\approx x$.

Now we state the propositions:

(105) dom(name-r(x)) = name-ord(x) = dom(name-y(x)). The theorem is a consequence of (104).

(106) name-r(x), name-y(x), name-ord(x) name like x. The theorem is a consequence of (105), (101), and (103).

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