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Suszko's Non-Fregean Logics. Part I[1]

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Summary. The basic properties of non-Fregean logics in general and of Sentential Calculus with Identity in particular, as introduced by Roman Suszko in [4] and [5].

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From now on k, m, n denote elements of $\mathbb{N}$, i, j denote natural numbers, a, b, c denote objects, y, z denote sets, and p, q, r, s denote finite sequences.

The functor VAR yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all $\langle 0, k \rangle$ where k is an element of $\mathbb{N}$.

Observe that VAR is non empty and antichain-like.

A variable is an element of VAR. The functors: *true*, *false*, 'not', &, and or yielding finite sequences are defined by terms

(Def. 2) $\langle 1 \rangle$,

(Def. 3) $\langle 2 \rangle$,

(Def. 4) $\langle 11 \rangle$,

(Def. 5) $\langle 21 \rangle$,

(Def. 6) $\langle 22 \rangle$,

respectively. The functors: SCI-unops and SCI-binops yielding non empty, finite sequence-membered sets are defined by terms

(Def. 7) $\{'\text{not}'\}$,

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(Def. 8) $\{\&, \mathrm{or}, '\mathrm{imp}', '\mathrm{eqv}', '{=}'\}$,

respectively. Now we state the proposition:

(1) (i) $a \in$ SCI-unops iff $a = '\mathrm{not}'$, and

(ii) $a \in$ SCI-binops iff $a = \&$ or $a = \mathrm{or}$ or $a = '\mathrm{imp}'$ or $a = '\mathrm{eqv}'$ or $a = '{=}'$.

Let F, G be non empty, finite sequence-membered sets. Observe that $F \cup G$ is non empty and finite sequence-membered.

The functor SCI-ops yielding a non empty, finite sequence-membered set is defined by the term

(Def. 9) SCI-unops $\cup$ SCI-binops.

Now we state the proposition:

(2) (i) if $p = \mathit{true}$, then $p(1) = 1$, and

(ii) if $p = \mathit{false}$, then $p(1) = 2$, and

(iii) if $p = '\mathrm{not}'$, then $p(1) = 11$, and

(iv) if $p = \&$, then $p(1) = 21$, and

(v) if $p = \mathrm{or}$, then $p(1) = 22$, and

(vi) if $p = '\mathrm{imp}'$, then $p(1) = 23$, and

(vii) if $p = '\mathrm{eqv}'$, then $p(1) = 24$, and

(viii) if $p = '{=}'$, then $p(1) = 25$.

Observe that SCI-ops is non empty and antichain-like.

The functor SCI-symbols yielding a non empty, finite sequence-membered set is defined by the term

(Def. 10) VAR $\cup$ SCI-ops.

The functors: VAR, SCI-ops, SCI-unops, and SCI-binops yield non empty subsets of SCI-symbols. The functors: $'\mathrm{not}'$, $\&$, or, $'\mathrm{imp}'$, and $'\mathrm{eqv}'$ yield elements of SCI-symbols. Note that SCI-symbols is non trivial and antichain-like.

One can verify that the functor SCI-symbols yields a non trivial Polish language. The functor SCI-op-arity yielding a function from SCI-ops into $\mathbb{N}$ is defined by the term

(Def. 11) (SCI-binops $\longmapsto 2) {+\cdot}$ (SCI-unops $\longmapsto 1)$.

The functor SCI-arity yielding a Polish arity-function of SCI-symbols is defined by the term

(Def. 12) SCI-op-arity $+\cdot$ (VAR $\longmapsto 0)$.

Now we state the propositions:

(3) If $a \in$ VAR, then (SCI-arity)$(a) = 0$.

(4) (i) (SCI-arity)('not') = 1, and

(ii) for every a such that $a \in$ SCI-binops holds (SCI-arity)(a) = 2.

(5) The Polish atoms(SCI-symbols , SCI-arity) = VAR. The theorem is a consequence of (4) and (3).

The functor SCI-formula-set yielding a full Polish language of SCI-symbols is defined by the term

(Def. 13) Polish-WFF-set(SCI-symbols, SCI-arity).

A SCI-formula is a Polish WFF of SCI-symbols and SCI-arity. Let us note that there exists a subset of SCI-formula-set which is non empty.

Let us consider n. The functor x_n yielding a SCI-formula is defined by the term

(Def. 14) $\langle 0, n \rangle$.

In the sequel X denotes an extension of SCI-arity, L denotes a Polish-ext-set of X, and t, u, v, w denote formulae of L.

Let us consider X. Now we state the propositions:

(6) SCI-symbols $\subseteq \mathrm{dom}\, X$.

(7) (i) 'not' $\in \mathrm{dom}\, X$, and

(ii) X('not') = 1, and

(iii) for every a such that $a \in$ SCI-binops holds $a \in \mathrm{dom}\, X$ and $X(a) = 2$. The theorem is a consequence of (6) and (4).

Let us consider X, L, and n. The functor $\mathrm{x}.(n, L)$ yielding a formula of L is defined by the term

(Def. 15) x_n.

Now we state the proposition:

(8) If $m \neq n$, then $\mathrm{x}_m \neq \mathrm{x}_n$.

Let us consider p. The functor $\neg p$ yielding a finite sequence is defined by the term

(Def. 16) 'not' $^\frown p$.

Let us consider q. The functors: $p \wedge q$, $p \vee q$, $p \Rightarrow q$, $p \Leftrightarrow q$, and $p{=}q$ yielding finite sequences are defined by terms

(Def. 17) & $^\frown (p {}^\frown q)$,

(Def. 18) or $^\frown (p {}^\frown q)$,

(Def. 19) 'imp' $^\frown (p {}^\frown q)$,

(Def. 20) 'eqv' $^\frown (p {}^\frown q)$,

(Def. 21) '=' $^\frown (p {}^\frown q)$, respectively. Let us consider X, L, and t. One can check that the functor $\neg t$ is defined by the term

(Def. 22) $(\text{Polish-unOp}(X, L, \text{'not'}))(t)$.

Let us consider u. The functors: $t \wedge u$, $t \vee u$, $t \Rightarrow u$, $t \Leftrightarrow u$, and $t{=}u$ are defined by terms

(Def. 23) $(\text{Polish-binOp}(X, L, \&))(t, u)$,

(Def. 24) $(\text{Polish-binOp}(X, L, \text{or}))(t, u)$,

(Def. 25) $(\text{Polish-binOp}(X, L, \text{'imp'}))(t, u)$,

(Def. 26) $(\text{Polish-binOp}(X, L, \text{'eqv'}))(t, u)$,

(Def. 27) $(\text{Polish-binOp}(X, L, \text{'='}))(t, u)$,

respectively. The functors: $\neg t$, $t \wedge u$, $t \vee u$, $t \Rightarrow u$, and $t \Leftrightarrow u$ yield formulae of L. The functor $t \Rightarrow u$ yielding a formula of L is defined by the term

(Def. 28) $t{=}(t \wedge u)$.

Let u be a SCI-formula. The functors: $t \Rightarrow u$ and $t{=}u$ yield formulae of L. The functors: $u \Rightarrow t$ and $u{=}t$ yield formulae of L. We say that t is atomic if and only if

(Def. 29) $t \in$ the Polish atoms(SCI-symbols , SCI-arity).

We say that t is negative if and only if

(Def. 30) $\text{PolishExtHead}(t) = \text{'not'}$.

We say that t is conjunctive if and only if

(Def. 31) $\text{PolishExtHead}(t) = \&$.

We say that t is disjunctive if and only if

(Def. 32) $\text{PolishExtHead}(t) = \text{or}$.

We say that t is conditional if and only if

(Def. 33) $\text{PolishExtHead}(t) = \text{'imp'}$.

We say that t is biconditional if and only if

(Def. 34) $\text{PolishExtHead}(t) = \text{'eqv'}$.

We say that t is an equality if and only if

(Def. 35) $\text{PolishExtHead}(t) = \text{'='}$.

Let us consider t. Now we state the propositions:

(9) t is atomic if and only if $t \in \text{VAR}$.

(10) t is negative if and only if there exists u such that $t = \neg u$.
PROOF: $\text{'not'} \in \text{dom}\, X$ and $X(\text{'not'}) = 1$. If t is negative, then there exists u such that $t = \neg u$ by [1, (9)]. □

(11) t is conjunctive if and only if there exists u and there exists v such that $t = u \wedge v$.

Proof: $\& \in \operatorname{dom} X$ and $X(\&) = 2$. If t is conjunctive, then there exists u and there exists v such that $t = u \wedge v$ by [1, (11)]. □

Let us consider X and L. The functors: SCI-prop-axioms(L) and SCI-id-axioms(L) yielding non empty subsets of L are defined by conditions

(Def. 36) for every a, $a \in$ SCI-prop-axioms(L) iff there exists t and there exists u and there exists v such that $a = t \Rightarrow u \Rightarrow t$ or $a = t \Rightarrow u \Rightarrow v \Rightarrow t \Rightarrow u \Rightarrow t \Rightarrow v$ or $a = \neg t \Rightarrow \neg u \Rightarrow u \Rightarrow t$ or $a = t \wedge u \Rightarrow \neg(t \Rightarrow \neg u)$ or $a = \neg(t \Rightarrow \neg u) \Rightarrow t \wedge u$ or $a = t \vee u \Rightarrow \neg t \Rightarrow u$ or $a = \neg t \Rightarrow u \Rightarrow t \vee u$ or $a = t \Leftrightarrow u \Rightarrow (t \Rightarrow u) \wedge (u \Rightarrow t)$ or $a = (t \Rightarrow u) \wedge (u \Rightarrow t) \Rightarrow t \Leftrightarrow u$,

(Def. 37) for every a, $a \in$ SCI-id-axioms(L) iff there exists t and there exists u and there exists v and there exists w such that $a = t{=}t$ or $a = t{=}u \Rightarrow (\neg t){=}(\neg u)$ or $a = t{=}u \wedge v{=}w \Rightarrow (t \wedge v){=}(u \wedge w)$ or $a = t{=}u \wedge v{=}w \Rightarrow (t \vee v){=}(u \vee w)$ or $a = t{=}u \wedge v{=}w \Rightarrow (t \Rightarrow v){=}(u \Rightarrow w)$ or $a = t{=}u \wedge v{=}w \Rightarrow (t \Leftrightarrow v){=}(u \Leftrightarrow w)$ or $a = t{=}u \wedge v{=}w \Rightarrow (t{=}v){=}(u{=}w)$ or $a = t{=}u \Rightarrow t \Rightarrow u$,

respectively. Let B be a subset of L. One can check that there exists a non empty subset of L which is B-extending.

The functor SCI-axioms(L) yielding a (SCI-prop-axioms(L))-extending subset of L is defined by the term

(Def. 38) SCI-prop-axioms(L) $\cup$ SCI-id-axioms(L).

From now on R, R_1, R_2 denote rules of L.

Let us consider X and L. The functor SCI-MP(L) yielding a rule of L is defined by the term

(Def. 39) the set of all $\langle\{t, t \Rightarrow u\}, u\rangle$ where t, u are formulae of L.

The functor SCI-rules(L) yielding a rule of L is defined by the term

(Def. 40) SCI-MP(L).

A formula-sequence of L is a finite sequence of elements of L.

A formula-finset of L is a finite subset of L. In the sequel A, A_1, A_2 denote non empty subsets of L, B, B_1, B_2 denote subsets of L, P, P_1, P_2 denote formula-sequences of L, and S, S_1, S_2 denote formula-finsets of L.

Let us consider X, L, and t. Let us note that the functor $\{t\}$ yields a formula-finset of L. Let us consider B and a. We say that a is B-provable if and only if

(Def. 41) a is (B, (SCI-rules(L)))-provable.

Note that every object which is B-axiomatic is also B-provable.

Now we state the proposition:

(12) If t is B-provable and $t \Rightarrow u$ is B-provable, then u is B-provable.

Let us consider X, L, and a. We say that a is L-prop-axiomatic if and only if

(Def. 42) a is (SCI-prop-axioms(L))-axiomatic.

We say that a is L-id-axiomatic if and only if

(Def. 43) a is (SCI-id-axioms(L))-axiomatic.

We say that a is L-SCI-axiomatic if and only if

(Def. 44) a is (SCI-axioms(L))-axiomatic.

We say that a is L-SCI-provable if and only if

(Def. 45) a is (SCI-axioms(L))-provable.

Observe that every element of SCI-prop-axioms(L) is L-prop-axiomatic and every element of SCI-id-axioms(L) is L-id-axiomatic and every element of SCI-axioms(L) is L-SCI-axiomatic and every object which is L-SCI-axiomatic is also L-SCI-provable and every object which is L-prop-axiomatic is also L-SCI-axiomatic and every object which is L-id-axiomatic is also L-SCI-axiomatic.

Let us consider t. Note that $t{=}t$ is L-id-axiomatic.

Let us consider u. Observe that $t \Rightarrow u \Rightarrow t$ is L-prop-axiomatic and $\neg t \Rightarrow \neg u \Rightarrow u \Rightarrow t$ is L-prop-axiomatic and $t \wedge u \Rightarrow \neg(t \Rightarrow \neg u)$ is L-prop-axiomatic and $\neg(t \Rightarrow \neg u) \Rightarrow t \wedge u$ is L-prop-axiomatic and $t \vee u \Rightarrow \neg t \Rightarrow u$ is L-prop-axiomatic and $\neg t \Rightarrow u \Rightarrow t \vee u$ is L-prop-axiomatic and $t \Leftrightarrow u \Rightarrow (t \Rightarrow u) \wedge (u \Rightarrow t)$ is L-prop-axiomatic and $(t \Rightarrow u) \wedge (u \Rightarrow t) \Rightarrow t \Leftrightarrow u$ is L-prop-axiomatic and $t{=}u \Rightarrow (\neg t){=}(\neg u)$ is L-id-axiomatic and $t{=}u \Rightarrow t \Rightarrow u$ is L-id-axiomatic.

Let us consider v. Note that $t \Rightarrow u \Rightarrow v \Rightarrow t \Rightarrow u \Rightarrow t \Rightarrow v$ is L-prop-axiomatic.

Let us consider w. Let O be an element of SCI-binops. One can verify that $t{=}u \wedge v{=}w \Rightarrow (\text{Polish-binOp}(X, L, O))(t, v){=}(\text{Polish-binOp}(X, L, O))(u, w)$ is L-id-axiomatic and there exists a formula of L which is L-prop-axiomatic and there exists a formula of L which is L-id-axiomatic.

In the sequel C denotes a (SCI-prop-axioms(L))-extending subset of L.

Let us consider X and L. One can check that every formula of L which is L-prop-axiomatic is also (SCI-prop-axioms(L))-provable.

Let us consider C. One can check that every formula of L which is non C-provable is also non (SCI-prop-axioms(L))-provable and there exists a formula of L which is C-provable.

Let us consider t. Let u be a C-provable formula of L. Note that $t \Rightarrow u$ is C-provable.

Now we state the propositions:

(13) If $t \Rightarrow u$ is C-provable and $u \Rightarrow v$ is C-provable, then $t \Rightarrow v$ is C-provable. The theorem is a consequence of (12).

(14) $t \Rightarrow t$ is C-provable. The theorem is a consequence of (12).

Let us consider X, L, and t. Let us note that $t \Rightarrow t$ is (SCI-prop-axioms(L))-provable.

Let us consider C. Let t be a C-provable formula of L. Let us consider u. Note that $t \Rightarrow u \Rightarrow u$ is C-provable.

Let us consider t. Let u be a C-provable formula of L. Let us consider v. One can verify that $t \Rightarrow u \Rightarrow v \Rightarrow t \Rightarrow v$ is C-provable.

Now we state the propositions:

(15) If $t \Rightarrow t \Rightarrow u$ is C-provable, then $t \Rightarrow u$ is C-provable. The theorem is a consequence of (12).

(16) If $t \Rightarrow u \Rightarrow v$ is C-provable, then $u \Rightarrow t \Rightarrow v$ is C-provable. The theorem is a consequence of (12) and (13).

Let us consider X, L, t, and u. Let us note that $t \Rightarrow t \Rightarrow u \Rightarrow t \Rightarrow u$ is (SCI-prop-axioms(L))-provable.

Let us consider X, L, C, and t. Now we state the propositions:

(17) $\neg\neg t \Rightarrow t$ is C-provable. The theorem is a consequence of (12).

(18) $t \Rightarrow \neg\neg t$ is C-provable. The theorem is a consequence of (17).

Let us consider X, L, and t. Note that $\neg\neg t \Rightarrow t$ is (SCI-prop-axioms(L))-provable and $t \Rightarrow \neg\neg t$ is (SCI-prop-axioms(L))-provable.

Let us consider u. One can check that $t \Rightarrow u \Rightarrow \neg u \Rightarrow \neg t$ is (SCI-prop-axioms(L))-provable.

Let us consider X, L, C, t, and u. Now we state the propositions:

(19) If $\neg t \Rightarrow u$ is C-provable, then $\neg u \Rightarrow t$ is C-provable. The theorem is a consequence of (13).

(20) If $t \Rightarrow \neg u$ is C-provable, then $u \Rightarrow \neg t$ is C-provable. The theorem is a consequence of (13).

(21) $\neg t \Rightarrow \neg u$ is C-provable if and only if $u \Rightarrow t$ is C-provable. The theorem is a consequence of (13).

Let us consider X, L, C, and t. Let u be a C-provable formula of L. Note that $\neg u \Rightarrow t$ is C-provable and $t \Rightarrow t$ is L-SCI-provable and $t \Rightarrow \neg\neg t$ is L-SCI-provable and $\neg\neg t \Rightarrow t$ is L-SCI-provable.

Let u be a L-SCI-provable formula of L. Note that $t \Rightarrow u$ is L-SCI-provable.

Now we state the proposition:

(22) $\neg t \Rightarrow t \Rightarrow u$ is C-provable.

Let us consider X, L, t, and u. Observe that $\neg t \Rightarrow t \Rightarrow u$ is (SCI-prop-axioms(L))-provable and $t \Rightarrow \neg t \Rightarrow u$ is (SCI-prop-axioms(L))-provable.

Now we state the proposition:

(23) If $\neg t$ is C-provable, then $t \Rightarrow u$ is C-provable.

Let us consider X, L, t, and u. One can check that $t \Rightarrow t \Rightarrow u \Rightarrow u$ is (SCI-prop-axioms(L))-provable.

Now we state the proposition:

(24) $t \Rightarrow u \Rightarrow v$ is C-provable if and only if $t \Rightarrow \neg v \Rightarrow \neg u$ is C-provable. The theorem is a consequence of (12), (13), and (16).

Let us consider X, L, C, t, and u. Now we state the propositions:

(25) (i) $t \wedge u \Rightarrow t$ is C-provable, and

(ii) $t \wedge u \Rightarrow u$ is C-provable.

The theorem is a consequence of (19) and (13).

(26) $t \Rightarrow u \Rightarrow t \wedge u$ is C-provable. The theorem is a consequence of (21), (13), (16), and (24).

Let us consider X, L, t, and u. Note that $t \wedge u \Rightarrow t$ is (SCI-prop-axioms(L))-provable and $t \wedge u \Rightarrow u$ is (SCI-prop-axioms(L))-provable and $t \Rightarrow u \Rightarrow t \wedge u$ is (SCI-prop-axioms(L))-provable.

Let us consider C. Let u be a C-provable formula of L. Observe that $t \Rightarrow t \wedge u$ is C-provable and $t \Rightarrow u \wedge t$ is C-provable.

Let t, u be C-provable formulae of L. One can check that $t \wedge u$ is C-provable.

Now we state the propositions:

(27) $t \wedge u \Rightarrow v$ is C-provable if and only if $t \Rightarrow u \Rightarrow v$ is C-provable. The theorem is a consequence of (12), (13), (16), and (15).

(28) $t \wedge u$ is C-provable if and only if t is C-provable and u is C-provable. The theorem is a consequence of (12).

(29) $t \Rightarrow u \wedge v$ is C-provable if and only if $t \Rightarrow u$ is C-provable and $t \Rightarrow v$ is C-provable. The theorem is a consequence of (13), (16), and (12).

(30) (i) $t \Rightarrow t \vee u$ is C-provable, and

(ii) $u \Rightarrow t \vee u$ is C-provable.

The theorem is a consequence of (13).

Let us consider X, L, and t. Let us note that $t \vee \neg t$ is (SCI-prop-axioms(L))-provable.

Let us consider u. One can verify that $t \Rightarrow t \vee u$ is (SCI-prop-axioms(L))-provable and $u \Rightarrow t \vee u$ is (SCI-prop-axioms(L))-provable.

Let us consider C. Let t be a C-provable formula of L. Let us observe that $t \vee u$ is C-provable and $u \vee t$ is C-provable.

Now we state the propositions:

(31) $\neg t \Rightarrow t \Rightarrow t$ is C-provable. The theorem is a consequence of (12) and (13).

(32) $t \vee u \Rightarrow v$ is C-provable if and only if $t \Rightarrow v$ is C-provable and $u \Rightarrow v$ is C-provable. The theorem is a consequence of (13), (21), and (31).

(33) Suppose $t \Rightarrow v$ is C-provable and $u \Rightarrow w$ is C-provable. Then

(i) $t \vee u \Rightarrow v \vee w$ is C-provable, and

(ii) $t \wedge u \Rightarrow v \wedge w$ is C-provable.

The theorem is a consequence of (13), (32), and (29).

(34) $t \Rightarrow u$ is C-provable if and only if u is $(C \cup \{t\})$-provable.

PROOF: Set $D = C \cup \{t\}$. If $t \Rightarrow u$ is C-provable, then u is D-provable by [2, (6)], (12). □

From now on D denotes a (SCI-axioms(L))-extending subset of L.

Let us consider X, L, and D. One can check that every formula of L which is non D-provable is also non L-SCI-provable and there exists a formula of L which is D-provable.

Let us consider X, L, D, t, and u. Now we state the propositions:

(35) If $t{=}u$ is D-provable, then $t \Rightarrow u$ is D-provable. The theorem is a consequence of (12).

(36) If $t{=}u$ is D-provable, then $(\neg t){=}(\neg u)$ is D-provable. The theorem is a consequence of (12).

(37) $t{=}u \Rightarrow u{=}t$ is D-provable. The theorem is a consequence of (13), (16), and (12).

(38) If $t{=}u$ is D-provable, then $u{=}t$ is D-provable. The theorem is a consequence of (37) and (12).

Now we state the propositions:

(39) $t{=}u \wedge v{=}u \Rightarrow t{=}v$ is D-provable. The theorem is a consequence of (37), (13), (16), and (12).

(40) If t is D-provable and $t \Rightarrow u$ is D-provable, then u is D-provable. The theorem is a consequence of (35), (12), and (28).

(41) Suppose $t{=}u$ is D-provable and $v{=}w$ is D-provable. Then

(i) $(t \wedge v){=}(u \wedge w)$ is D-provable, and

(ii) $t \wedge v \Rightarrow u \wedge w$ is D-provable, and

(iii) if $t \wedge v$ is D-provable, then $u \wedge w$ is D-provable, and

(iv) $(t \vee v){=}(u \vee w)$ is D-provable, and

(v) $t \vee v \Rightarrow u \vee w$ is D-provable, and

(vi) if $t \vee v$ is D-provable, then $u \vee w$ is D-provable, and

(vii) $(t \Rightarrow v){=}(u \Rightarrow w)$ is D-provable, and

(viii) $t \Rightarrow v \Rightarrow u \Rightarrow w$ is D-provable, and

(ix) if $t \Rightarrow v$ is D-provable, then $u \Rightarrow w$ is D-provable, and

(x) $(t \Leftrightarrow v){=}(u \Leftrightarrow w)$ is D-provable, and

(xi) $t \Leftrightarrow v \Rightarrow u \Leftrightarrow w$ is D-provable, and

(xii) if $t \Leftrightarrow v$ is D-provable, then $u \Leftrightarrow w$ is D-provable, and

(xiii) $(t{=}v){=}(u{=}w)$ is D-provable, and

(xiv) $t{=}v \Rightarrow u{=}w$ is D-provable, and

(xv) if $t{=}v$ is D-provable, then $u{=}w$ is D-provable, and

(xvi) $(t \Rightarrow v){=}(u \Rightarrow w)$ is D-provable, and

(xvii) $t \Rightarrow v \Rightarrow u \Rightarrow w$ is D-provable, and

(xviii) if $t \Rightarrow v$ is D-provable, then $u \Rightarrow w$ is D-provable.

The theorem is a consequence of (12) and (35).

Let us consider X, L, D, t, u, and v. Now we state the propositions:

(42) (i) $t{=}u \Rightarrow (t \wedge v){=}(u \wedge v)$ is D-provable, and

(ii) $t{=}u \Rightarrow (v \wedge t){=}(v \wedge u)$ is D-provable, and

(iii) $t{=}u \Rightarrow (t \vee v){=}(u \vee v)$ is D-provable, and

(iv) $t{=}u \Rightarrow (v \vee t){=}(v \vee u)$ is D-provable, and

(v) $t{=}u \Rightarrow (t \Rightarrow v){=}(u \Rightarrow v)$ is D-provable, and

(vi) $t{=}u \Rightarrow (v \Rightarrow t){=}(v \Rightarrow u)$ is D-provable, and

(vii) $t{=}u \Rightarrow (t \Leftrightarrow v){=}(u \Leftrightarrow v)$ is D-provable, and

(viii) $t{=}u \Rightarrow (v \Leftrightarrow t){=}(v \Leftrightarrow u)$ is D-provable, and

(ix) $t{=}u \Rightarrow (t{=}v){=}(u{=}v)$ is D-provable, and

(x) $t{=}u \Rightarrow (v{=}t){=}(v{=}u)$ is D-provable, and

(xi) $t{=}u \Rightarrow (t \Rightarrow v){=}(u \Rightarrow v)$ is D-provable, and

(xii) $t{=}u \Rightarrow (v \Rightarrow t){=}(v \Rightarrow u)$ is D-provable.

The theorem is a consequence of (13) and (29).

(43) Suppose $t{=}u$ is D-provable. Then

(i) t is D-provable iff u is D-provable, and

(ii) $(t \wedge v){=}(u \wedge v)$ is D-provable, and

(iii) $t \wedge v \Rightarrow u \wedge v$ is D-provable, and

(iv) if $t \wedge v$ is D-provable, then $u \wedge v$ is D-provable, and

(v) $(v \wedge t){=}(v \wedge u)$ is D-provable, and

(vi) $v \wedge t \Rightarrow v \wedge u$ is D-provable, and

(vii) if $v \wedge t$ is D-provable, then $v \wedge u$ is D-provable, and

(viii) $(t \vee v){=}(u \vee v)$ is D-provable, and

(ix) $t \vee v \Rightarrow u \vee v$ is D-provable, and

(x) if $t \vee v$ is D-provable, then $u \vee v$ is D-provable, and

(xi) $(v \vee t)=(v \vee u)$ is D-provable, and

(xii) $v \vee t \Rightarrow v \vee u$ is D-provable, and

(xiii) if $v \vee t$ is D-provable, then $v \vee u$ is D-provable, and

(xiv) $(t \Rightarrow v)=(u \Rightarrow v)$ is D-provable, and

(xv) $t \Rightarrow v \Rightarrow u \Rightarrow v$ is D-provable, and

(xvi) if $t \Rightarrow v$ is D-provable, then $u \Rightarrow v$ is D-provable, and

(xvii) $(v \Rightarrow t)=(v \Rightarrow u)$ is D-provable, and

(xviii) $v \Rightarrow t \Rightarrow v \Rightarrow u$ is D-provable, and

(xix) if $v \Rightarrow t$ is D-provable, then $v \Rightarrow u$ is D-provable, and

(xx) $(t \Leftrightarrow v)=(u \Leftrightarrow v)$ is D-provable, and

(xxi) $t \Leftrightarrow v \Rightarrow u \Leftrightarrow v$ is D-provable, and

(xxii) if $t \Leftrightarrow v$ is D-provable, then $u \Leftrightarrow v$ is D-provable, and

(xxiii) $(v \Leftrightarrow t)=(v \Leftrightarrow u)$ is D-provable, and

(xxiv) $v \Leftrightarrow t \Rightarrow v \Leftrightarrow u$ is D-provable, and

(xxv) if $v \Leftrightarrow t$ is D-provable, then $v \Leftrightarrow u$ is D-provable, and

(xxvi) $(t{=}v)=(u{=}v)$ is D-provable, and

(xxvii) $t{=}v \Rightarrow u{=}v$ is D-provable, and

(xxviii) if $t{=}v$ is D-provable, then $u{=}v$ is D-provable, and

(xxix) $(v{=}t)=(v{=}u)$ is D-provable, and

(xxx) $v{=}t \Rightarrow v{=}u$ is D-provable, and

(xxxi) if $v{=}t$ is D-provable, then $v{=}u$ is D-provable, and

(xxxii) $(t \Rightarrow v)=(u \Rightarrow v)$ is D-provable, and

(xxxiii) $t \Rightarrow v \Rightarrow u \Rightarrow v$ is D-provable, and

(xxxiv) if $t \Rightarrow v$ is D-provable, then $u \Rightarrow v$ is D-provable, and

(xxxv) $(v \Rightarrow t)=(v \Rightarrow u)$ is D-provable, and

(xxxvi) $v \Rightarrow t \Rightarrow v \Rightarrow u$ is D-provable, and

(xxxvii) if $v \Rightarrow t$ is D-provable, then $v \Rightarrow u$ is D-provable.

The theorem is a consequence of (38), (35), (12), and (42).

Let us consider X and L.

A congruence of L is an equivalence relation of L defined by

(Def. 46) for every t, u, v, and w such that $\langle t, u\rangle$, $\langle v, w\rangle \in it$ holds $\langle \neg t, \neg u\rangle$, $\langle t \wedge v, u \wedge w\rangle$, $\langle t \vee v, u \vee w\rangle$, $\langle t \Rightarrow v, u \Rightarrow w\rangle$, $\langle t \Leftrightarrow v, u \Leftrightarrow w\rangle$, $\langle t{=}v, u{=}w\rangle \in it$.

In the sequel E denotes a congruence of L.

Let us consider X and L. Let us observe that there exists a family of subsets of L which is non empty.

Let us consider E.

A □-equivalence class of E is an element of Classes E. Let us consider t. The functor E-class t yielding a □-equivalence class of E is defined by the term

(Def. 47) $[t]_E$.

Now we state the proposition:

(44) $\langle t, u\rangle \in E$ if and only if E-class $t = E$-class u.

PROOF: If $\langle t, u\rangle \in E$, then E-class $t = E$-class u by [3, (18), (23)]. □

From now on d, e denote □-equivalence classes of E.

Now we state the proposition:

(45) There exists t such that $d = E$-class t.

Let us consider X, L, E, and d. The functor $\neg d$ yielding a □-equivalence class of E is defined by

(Def. 48) there exists t such that $d = E$-class t and $it = E$-class $\neg t$.

Let us consider e. The functors: $d \wedge e$, $d \vee e$, $d \Rightarrow e$, and $d \Leftrightarrow e$ yielding □-equivalence classes of E are defined by conditions

(Def. 49) there exists t and there exists u such that $d = E$-class t and $e = E$-class u and $d \wedge e = E$-class $t \wedge u$,

(Def. 50) there exists t and there exists u such that $d = E$-class t and $e = E$-class u and $d \vee e = E$-class$(t \vee u)$,

(Def. 51) there exists t and there exists u such that $d = E$-class t and $e = E$-class u and $d \Rightarrow e = E$-class$(t \Rightarrow u)$,

(Def. 52) there exists t and there exists u such that $d = E$-class t and $e = E$-class u and $d \Leftrightarrow e = E$-class$(t \Leftrightarrow u)$,

respectively. Let us consider D. The functor EqRel(D) yielding a congruence of L is defined by

(Def. 53) for every t and u, $\langle t, u\rangle \in it$ iff $t{=}u$ is D-provable.

A □-equivalence class of D is a □-equivalence class of EqRel(D). Let us consider t. The functor D-class t yielding a □-equivalence class of D is defined by the term

(Def. 54) EqRel(D)-class t.

Now we state the proposition:

(46) $t{=}u$ is D-provable if and only if D-class $t = D$-class u. The theorem is a consequence of (44).

In the sequel x, y, z denote $\Box$-equivalence classes of D.

Now we state the proposition:

(47) There exists t such that $x = D$-class t. The theorem is a consequence of (45).

Let us consider X, L, D, and x. We say that x is D-provable if and only if

(Def. 55) there exists t such that $x = D$-class t and t is D-provable.

Now we state the proposition:

(48) $y = \neg x$ if and only if there exists t such that $x = D$-class t and $y = D$-class $\neg t$.

Let us consider X, L, and D. Let t be a D-provable formula of L. Note that D-class t is D-provable and there exists a $\Box$-equivalence class of D which is D-provable.

Now we state the proposition:

(49) If D-class t is D-provable, then t is D-provable. The theorem is a consequence of (46), (35), and (12).

Let us consider X, L, and D. Let x be a D-provable $\Box$-equivalence class of D. Let us observe that every element of x is D-provable.

Let us consider x and y. Now we state the propositions:

(50) $x \wedge y$ is D-provable if and only if x is D-provable and y is D-provable. The theorem is a consequence of (47), (49), and (28).

(51) $x{=}y$ is D-provable if and only if $x = y$. The theorem is a consequence of (47), (46), and (49).

Now we state the propositions:

(52) (i) D-class $\neg t = \neg(D$-class $t)$, and

(ii) D-class $t \wedge u = (D$-class $t) \wedge (D$-class $u)$, and

(iii) D-class$(t \vee u) = (D$-class $t) \vee (D$-class $u)$, and

(iv) D-class$(t \Rightarrow u) = D$-class $t \Rightarrow D$-class u, and

(v) D-class$(t \Leftrightarrow u) = D$-class $t \Leftrightarrow D$-class u, and

(vi) D-class $t{=}u = (D$-class $t){=}(D$-class $u)$.

(53) If x is D-provable, then $x \vee y$ is D-provable and $y \vee x$ is D-provable. The theorem is a consequence of (47) and (49).

Let us consider X, L, D, t, and u. Now we state the propositions:

(54) $t \Leftrightarrow u$ is D-provable if and only if $t \Rightarrow u$ is D-provable and $u \Rightarrow t$ is D-provable. The theorem is a consequence of (12) and (28).

(55) If $t \Leftrightarrow u$ is D-provable, then $u \Leftrightarrow t$ is D-provable. The theorem is a consequence of (54).

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