

The Ackermann Function and Primitive Recursive Functions

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Summary. This paper formalizes the classical proof that the Ackermann function is not primitive recursive. The intermediate results establish the recursive equations, monotonicity, and growth estimates needed to show that every primitive recursive function is majorized by a suitable level of the Ackermann function. The final diagonal argument then proves that the binary Ackermann function cannot itself be primitive recursive.

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1. PRELIMINARIES

From now on D denotes a set and $a, b, i, k, m, n, n_1, n_2$ denote natural numbers.

Now we state the proposition:

$$(1) \quad m \uparrow\uparrow(n+1) = m^{m \uparrow\uparrow n}.$$

Let us consider m . Observe that $m \uparrow\uparrow 1$ reduces to m .

Now we state the propositions:

$$(2) \quad m \uparrow\uparrow 2 = m^m.$$

$$(3) \quad m \uparrow\uparrow 3 = m^{m^m}.$$

$$(4) \quad m \uparrow\uparrow 4 = m^{m^{m^m}}.$$

$$(5) \quad m \uparrow\uparrow 5 = m^{m^{m^{m^m}}}.$$

2. ACKERMANN FUNCTION

The functor **AckermannFunction** yielding a binary operation on $\mathbb{N}$ is defined by

- (Def. 1) for every natural numbers m, n , $it(0, n) = n + 1$ and if $m > 0$, then $it(m, 0) = it(m - 1, 1)$ and if $m > 0$ and $n > 0$, then $it(m, n) = it(m - 1, it(m, n - 1))$.

Let m, n be natural numbers. The functor **Ackermann(m, n)** yielding an element of $\mathbb{N}$ is defined by the term

- (Def. 2) $(\text{AckermannFunction})(m, n)$.

Now we state the propositions:

- (6) $\text{Ackermann}(m + 1, 0) = \text{Ackermann}(m, 1)$.
- (7) $\text{Ackermann}(m + 1, n + 1) = \text{Ackermann}(m, \text{Ackermann}(m + 1, n))$.
- (8) $\text{Ackermann}(0, n) = n + 1$.
- (9) $\text{Ackermann}(1, n) = n + 2$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathcal{A}(1, \$_1) = \$_1 + 2$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (10) $\text{Ackermann}(2, n) = 2 \cdot n + 3$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathcal{A}(2, \$_1) = 2 \cdot \$_1 + 3$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (11) $\text{Ackermann}(3, n) = 2^{n+3} - 3$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathcal{A}(3, \$_1) = 2^{\$1+3} - 3$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by (10), [6, (6)]. For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (12) $\text{Ackermann}(4, n) = 2 \uparrow \uparrow (n + 3) - 3$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathcal{A}(4, \$_1) = 2 \uparrow \uparrow (\$1 + 3) - 3$. $\mathcal{P}[0]$ by [2, (19)], (11). For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [1, (38)], [2, (14)], (11). For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (13) $\text{Ackermann}(4, 0) = 13$. The theorem is a consequence of (3) and (12).

- (14) $\text{Ackermann}(4, 1) = 65533$. The theorem is a consequence of (4) and (12).

- (15) $\text{Ackermann}(4, 2) = 2^{65536} - 3$. The theorem is a consequence of (5) and (12).

- (16) $n < \text{Ackermann}(m, n)$.

PROOF: Define $\mathcal{M}[\text{natural number}] \equiv \text{for every } n, n < \mathcal{A}(\$1, n)$. $\mathcal{M}[0]$ by [1, (16)]. For every m such that $\mathcal{M}[m]$ holds $\mathcal{M}[m + 1]$ by (6), [1, (13)], (7). For every k , $\mathcal{M}[k]$ from [1, Sch. 2]. $\square$

- (17) $\text{Ackermann}(m, n) < \text{Ackermann}(m, n + 1)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathcal{A}(\$1, n) < \mathcal{A}(\$1, n+1)$. $\mathcal{P}[0]$ by [1, (16)]. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (18) If $a < b$, then $\text{Ackermann}(m, a) < \text{Ackermann}(m, b)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{for every } a \text{ such that } a < \$1 \text{ holds } \mathcal{A}(m, a) < \mathcal{A}(m, \$1)$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [1, (13)], (17). For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (19) If $a \leq b$, then $\text{Ackermann}(m, a) \leq \text{Ackermann}(m, b)$. The theorem is a consequence of (18).

- (20) $m < \text{Ackermann}(m, n)$.

PROOF: Define $\mathcal{M}[\text{natural number}] \equiv \text{for every } m, m < \mathcal{A}(m, \$1)$. $\mathcal{M}[0]$ by [1, (13)], (17), (6). For every n such that $\mathcal{M}[n]$ holds $\mathcal{M}[n+1]$. For every k , $\mathcal{M}[k]$ from [1, Sch. 2]. $\square$

- (21) $\text{Ackermann}(m, n+1) \leq \text{Ackermann}(m+1, n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{for every } m, \mathcal{A}(m, \$1+1) \leq \mathcal{A}(m+1, \$1)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (16), [1, (13)], (19), (7). For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (22) $\text{Ackermann}(m, n) < \text{Ackermann}(m+1, n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{for every } n, \mathcal{A}(\$1, n) < \mathcal{A}(\$1+1, n)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (23) If $a < b$, then $\text{Ackermann}(a, n) < \text{Ackermann}(b, n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{for every } a \text{ such that } a < \$1 \text{ holds } \mathcal{A}(a, n) < \mathcal{A}(\$1, n)$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [1, (13)], (22). For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (24) If $a \leq b$, then $\text{Ackermann}(a, n) \leq \text{Ackermann}(b, n)$. The theorem is a consequence of (23).

- (25) $\text{Ackermann}(k, \text{Ackermann}(m, n)) < \text{Ackermann}(k+m+2, n)$. The theorem is a consequence of (24), (23), (18), (7), and (21).

- (26) Let us consider natural numbers r, s, y . Then $\text{Ackermann}(r, y) + \text{Ackermann}(s, y) < \text{Ackermann}(4 + \max(r, s), y)$. The theorem is a consequence of (24), (10), and (25).

- (27) Let us consider natural numbers r, n . Then $\text{Ackermann}(r, n+n) < \text{Ackermann}(r+4, n)$.

PROOF: $n+n < \mathcal{A}(2, n)$ by (10), [1, (16)]. $\mathcal{A}(r, n+n) < \mathcal{A}(r, \mathcal{A}(2, n))$. $\mathcal{A}(r, n+n) < \mathcal{A}(r+2+2, n)$. $\square$

Let us consider m and n . One can verify that $\text{Ackermann}(m, n)$ is non zero.

The functor **AckermannFunction(m)** yielding a unary operation on $\mathbb{N}$ is

defined by

(Def. 3) for every natural number n , $it(n) = \text{Ackermann}(m, n)$.

Observe that $\text{AckermannFunction}(m)$ is positive yielding and $\text{AckermannFunction}(m)$ is increasing.

Now we state the propositions:

(28) $(\text{AckermannFunction}(m+1))(0) = (\text{AckermannFunction}(m))(1)$. The theorem is a consequence of (6).

(29) $(\text{AckermannFunction}(m+1))(n+1) = (\text{AckermannFunction}(m))((\text{AckermannFunction}(m+1))(n))$. The theorem is a consequence of (7).

3. AUXILIARY OPERATIONS

Let f be a function. The functor $\text{dom2tuple}(f)$ yielding a function is defined by

(Def. 4) (i) $\text{dom } it = \text{dom } f^1$ and for every object x such that $x \in \text{dom } f$ holds $it(\langle x \rangle) = f(x)$, **if** f is not empty,
(ii) $it = \emptyset$, **otherwise**.

Let f be an empty function. One can check that $\text{dom2tuple}(f)$ is empty.

Let f be a non empty function. One can check that $\text{dom2tuple}(f)$ is non empty.

Let D be a set and f be a D -defined function. One can check that $\text{dom2tuple}(f)$ is (D^*) -defined.

Let f be a D -valued function. One can verify that $\text{dom2tuple}(f)$ is D -valued.

Let D be a non empty set and f be a D -defined function. Observe that $\text{dom2tuple}(f)$ is homogeneous.

Now we state the proposition:

(30) Let us consider a unary operation f on D . Then $\text{dom2tuple}(f)$ is a partial function from D^* to D .

Let D be a non empty set and f be a unary operation on D . Let us observe that $\text{dom2tuple}(f)$ is 1-ary.

Let D be a set and f be a binary operation on D . The functor $\text{dom2tuple}(f)$ yielding a function is defined by

(Def. 5) (i) $\text{dom } it = D^2$ and for every objects x, y such that $\langle x, y \rangle \in \text{dom } f$ holds $it(\langle x, y \rangle) = f(x, y)$, **if** D is not empty,
(ii) $it = \emptyset$, **otherwise**.

Let f be a binary operation on $\mathbb{N}$. Observe that $\text{dom2tuple}(f)$ is $(\mathbb{N}^*)$ -defined.

Let D be a set and f be a binary operation on D . One can verify that $\text{dom2tuple}(f)$ is D -valued.

Now we state the proposition:

- (31) Let us consider a binary operation f on D . Then $\text{dom2tuple}(f)$ is a partial function from D^* to D .

Let D be a non empty set and f be a binary operation on D . Observe that $\text{dom2tuple}(f)$ is homogeneous and $\text{dom2tuple}(f)$ is 2-ary.

Now we state the proposition:

- (32) $\text{dom2tuple}(\text{AckermannFunction}) \in \text{HFuncs } \mathbb{N}$.

4. PRIMITIVE RECURSIVENESS

Let f be a function yielding many sorted set indexed by $\mathbb{N}$. We say that f is homogeneous if and only if

- (Def. 6) for every natural number n , $f(n)$ is homogeneous.

We say that f into $\mathbb{N}$ if and only if

- (Def. 7) for every natural number n , $f(n)$ inherits into $\mathbb{N}$.

We say that f is primitive recursive if and only if

- (Def. 8) for every natural number n , $f(n)$ is primitive recursive.

Let D be a set. We say that f is D -defined if and only if

- (Def. 9) for every natural number n , $f(n)$ is D -defined.

We say that f is satisfying primrec conditions if and only if

- (Def. 10) f is homogeneous and $(\mathbb{N}^*)$ -defined.

Let f be a homogeneous function. One can verify that $\mathbb{N} \mapsto f$ is homogeneous.

Let f be a function. One can check that $\mathbb{N} \mapsto f$.

Let f be a primitive recursive function. Note that $\mathbb{N} \mapsto f$ is primitive recursive.

Let D be a set and f be a D -defined function. Note that $\mathbb{N} \mapsto f$ is D -defined and there exists a function yielding many sorted set indexed by $\mathbb{N}$ which is satisfying primrec conditions and every function yielding many sorted set indexed by $\mathbb{N}$ which is satisfying primrec conditions is also homogeneous and $(\mathbb{N}^*)$ -defined and every function yielding many sorted set indexed by $\mathbb{N}$ which is homogeneous and $(\mathbb{N}^*)$ -defined is also satisfying primrec conditions.

Let f be a homogeneous, function yielding many sorted set indexed by $\mathbb{N}$ and n be a natural number. Note that $f(n)$ is homogeneous.

Let f be a function yielding many sorted set indexed by $\mathbb{N}$. Let us note that $f(n)$ and there exists a function yielding many sorted set indexed by $\mathbb{N}$ which is primitive recursive.

Let f be a primitive recursive, function yielding many sorted set indexed by $\mathbb{N}$ and n be a natural number. Let us observe that $f(n)$ is primitive recursive.

Let D be a set. Observe that there exists a function yielding many sorted set indexed by $\mathbb{N}$ which is D -defined.

Let f be a D -defined, function yielding many sorted set indexed by $\mathbb{N}$ and n be a natural number. Observe that $f(n)$ is D -defined.

Let k be a natural number and f be a homogeneous, function yielding many sorted set indexed by $\mathbb{N}$. We say that f is k -ary if and only if

(Def. 11) for every natural number n , $f(n)$ is k -ary.

Let k be a non zero natural number. Let us observe that there exists a homogeneous, function yielding many sorted set indexed by $\mathbb{N}$ which is k -ary, primitive recursive, and $(\mathbb{N}^*)$ -defined.

Let f be a k -ary, homogeneous, function yielding many sorted set indexed by $\mathbb{N}$ and n be a natural number. Let us observe that $f(n)$ is k -ary.

Let k be a natural number. The functor **k -aryPrimRecFuncs** yielding a subset of $\mathbf{HFuncs} \mathbb{N}$ is defined by the term

(Def. 12) $\{f, \text{ where } f \text{ is an element of } \mathbf{HFuncs} \mathbb{N} : f \text{ is } k\text{-ary and primitive recursive}\}.$

Now we state the proposition:

(33) $\text{const}_k(n) \in k\text{-aryPrimRecFuncs}.$

Let k be a natural number. Let us note that $k\text{-aryPrimRecFuncs}$ is non empty.

Now we state the proposition:

(34) If $1 \leq n \leq k$, then $\text{succ}_k(n) \in k\text{-aryPrimRecFuncs}.$

PROOF: Set $f = \text{succ}_k(n)$. f is k -ary. f is primitive recursive by [4, (70)].

□

The functor **AckermannSequence** yielding a 1-ary, satisfying primrec conditions, primitive recursive, function yielding many sorted set indexed by $\mathbb{N}$ is defined by

(Def. 13) $it(0) = \text{succ}_1(1)$ and for every natural number m , $it(m+1) = \text{primrec}(\text{const}_0(\text{AckermannSequence}(\prod^* \langle \text{proj}_2(2) \rangle), 1)).$

Note that **AckermannSequence** is $(\mathbf{HFuncs} \mathbb{N})$ -valued.

Now we state the propositions:

(35) $\text{arity}(\text{AckermannSequence})(m) \cdot (\prod^* \langle \text{proj}_2(2) \rangle) = 2.$

(36) $(\text{AckermannSequence})(m)(\langle n \rangle) = \text{Ackermann}(m, n).$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{for every } n, AS(\$_1)(\langle n \rangle) = \mathcal{A}(\$_1, n).$ $\mathcal{P}[0]$ by [5, (98)]. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (80)], [7, (7)], (6), [4, (36)]. For every k , $\mathcal{P}[k]$ from [1, Sch. 2]. □

(37) $\text{dom2tuple}(\text{AckermannFunction}(0)) = \text{succ}_1(1)$.

(38) $\text{dom2tuple}(\text{AckermannFunction}(m+1)) = \text{primrec}(\text{const}_0(\text{Ackermann}(m, 1)), (\text{Ack}(\prod^*(\text{proj}_2(2))), 1)$. The theorem is a consequence of (35) and (36).

(39) $\text{dom2tuple}(\text{AckermannFunction}(m)) = (\text{AckermannSequence})(m)$. The theorem is a consequence of (37) and (38).

Let us consider m . Let us note that $\text{dom2tuple}(\text{AckermannFunction}(m))$ is primitive recursive.

Let k be a natural number. Observe that there exists a homogeneous, $(\mathbb{N}^*)$ -defined function which is k -ary.

Let f be a binary operation on $\mathbb{N}$ and g be a function. We say that f majorizes g if and only if

(Def. 14) there exists an element N of $\mathbb{N}$ such that for every non empty finite sequence F of elements of $\mathbb{N}$ such that $F \in \text{dom } g$ for every element a of $\mathbb{N}$ such that $a = \max \text{rng } F$ holds $g(F) < f(N, a)$.

Now we state the proposition:

(40) Let us consider a binary operation f on $\mathbb{N}$. Then f does not majorize $\text{dom2tuple}(f)$.

Let f be a binary operation on $\mathbb{N}$. The functor $\text{majorizedBy}(f)$ yielding a subset of $\text{HFuncs } \mathbb{N}$ is defined by the term

(Def. 15) $\{g, \text{ where } g \text{ is a homogeneous, } (\mathbb{N}^*)\text{-defined function} : f \text{ majorizes } g\}$.

Now we state the propositions:

(41) Let us consider a homogeneous, $(\mathbb{N}^*)$ -defined function g . If $\text{arity } g = 0$, then AckermannFunction majorizes g .

(42) Let us consider a finite sequence G of elements of $\text{HFuncs } \mathbb{N}$. Suppose $\text{rng } G \subseteq \text{majorizedBy}(\text{AckermannFunction})$. Then there exists an element N of $\mathbb{N}$ such that for every homogeneous, $(\mathbb{N}^*)$ -defined function g such that $g \in \text{rng } G$ for every non empty finite sequence F of elements of $\mathbb{N}$ such that $F \in \text{dom } g$ for every element a of $\mathbb{N}$ such that $a = \max \text{rng } F$ holds $g(F) < \text{Ackermann}(N, a)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence G of elements of $\text{HFuncs } \mathbb{N}$ such that $\text{len } G = \$_1$ and $\text{rng } G \subseteq \text{majorizedBy}(\text{AckermannFunction})$ there exists an element N of $\mathbb{N}$ such that for every homogeneous, $(\mathbb{N}^*)$ -defined function g such that $g \in \text{rng } G$ for every non empty finite sequence F of elements of $\mathbb{N}$ such that $F \in \text{dom } g$ for every element a of $\mathbb{N}$ such that $a = \max \text{rng } F$ holds $g(F) < \text{Ackermann}(N, a)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (18)], [8, (53)], [9, (70)], [1, (11)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

- (43) AckermannFunction majorizes $\text{const}_m(n)$. The theorem is a consequence of (20).
- (44) $\text{const}_m(n) \in \text{majorizedBy}(\text{AckermannFunction})$. The theorem is a consequence of (43).
- (45) If $\text{const}_m(n_1) = \text{const}_m(n_2)$, then $n_1 = n_2$.
- (46) If $1 \leq n_1 \leq m$ and $1 \leq n_2 \leq m$ and $\text{succ}_m(n_1) = \text{succ}_m(n_2)$, then $n_1 = n_2$.
- (47) If $1 \leq n_1 \leq m$ and $1 \leq n_2 \leq m$ and $\text{proj}_m(n_1) = \text{proj}_m(n_2)$, then $n_1 = n_2$.

Let m be a natural number. The functor $\text{constF}(m)$ yielding a function from $\mathbb{N}$ into $\text{HFuncs } \mathbb{N}$ is defined by

(Def. 16) for every natural number n , $it(n) = \text{const}_m(n)$.

Let us consider m . One can check that $\text{constF}(m)$ is one-to-one.

Now we state the proposition:

- (48) $\text{rng } \text{constF}(m) \subseteq \text{PrimRec}$.

Let us note that PrimRec is infinite.

Now we state the propositions:

- (49) If $1 \leq n \leq m$, then AckermannFunction majorizes $\text{succ}_m(n)$. The theorem is a consequence of (9) and (24).
- (50) If $1 \leq n \leq m$, then $\text{succ}_m(n) \in \text{majorizedBy}(\text{AckermannFunction})$. The theorem is a consequence of (49).
- (51) If $1 \leq n \leq m$, then AckermannFunction majorizes $\text{proj}_m(n)$. The theorem is a consequence of (24) and (16).
- (52) If $1 \leq n \leq m$, then $\text{proj}_m(n) \in \text{majorizedBy}(\text{AckermannFunction})$. The theorem is a consequence of (51).

Let us observe that $\text{majorizedBy}(\text{AckermannFunction})$ is composition closed and $\text{majorizedBy}(\text{AckermannFunction})$ is primitive recursion closed and $\text{majorizedBy}(\text{AckermannFunction})$ is primitive recursively closed.

Now we state the proposition:

- (53) $\text{PrimRec} \subseteq \text{majorizedBy}(\text{AckermannFunction})$.

Let us note that $\text{majorizedBy}(\text{AckermannFunction})$ is infinite.

Now we state the propositions:

- (54) $\text{dom2tuple}(\text{AckermannFunction})$ is not primitive recursive. The theorem is a consequence of (53) and (40).
- (55) $\text{dom2tuple}(\text{AckermannFunction}) \notin \text{PrimRec}$.
- (56) Let us consider a unary operation d on $\mathbb{N}$. Suppose for every natural number n , $d(n) = \text{Ackermann}(n, n)$. Then $\text{dom2tuple}(d)$ is not primitive

recursive. The theorem is a consequence of (53).

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. Epsilon numbers and Cantor normal form. *Formalized Mathematics*, 17(4):249–256, 2009. doi:10.2478/v10037-009-0032-8.
- [3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [4] Grzegorz Bancerek and Piotr Rudnicki. The set of primitive recursive functions. *Formalized Mathematics*, 9(4):705–720, 2001.
- [5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [6] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(5):887–890, 1990.
- [7] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [8] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(3):569–573, 1990.
- [9] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

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