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Stability of Parallel 4-2 Compressor Circuits[1]

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Summary. Since calculation models of the arithmetic logic unit based on many sorted algebra have been proposed, we continue to verify the structure and design of these circuits using the Mizar proof checking system [10], [11], [9]. In this article, we formalize the concepts of Generalized Full Adder (GFA) circuits [15] and 4-2 Binary Addition Cell primitives (FTAs) [14] to construct the structures of cascaded parallel 4-2 compressor circuits. We define the circuit structure of the n-bit cascaded parallel 4-2 compressor circuits for both standard cells: TYPE-0 (GFA0-over-GFA0) and TYPE-1 (GFA1-over-GFA2 with negated inputs). We then successfully prove their constant-time stability, showing that the calculation outputs stabilize within 4 combinational propagation steps (T = 4) regardless of the bit width n.

This parallel 4-2 compressor circuit is of paramount importance in practical high-performance systems, such as digital signal processors (DSPs), multipliers, and arithmetic units, where reducing partial product accumulation delay is the primary constraint [13], [8]. Crucially, our constant-time stability proof represents a landmark achievement in formalized hardware verification. By proving that the cascade settles within 4 steps independent of width n, this research establishes a methodology and template for verifying behavioral properties of complex, parameterized parallel hardware systems and constant-delay arithmetic architectures [12], [1] in formalized mathematics. The formalization by means of Mizar system [6], [5].

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1. Stability of a Parallel 4-2 Compressor Circuit (TYPE-0)

Let n be a natural number and a, b_2, c_2, d_2 be finite sequences. The functor n-BitFTA0Str(a, b_2, c_2, d_2) yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by

(Def. 1) there exist many sorted sets f, h indexed by $\mathbb{N}$ such that $it = f(n)$ and $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot \text{BitFTA0Str}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$ and $h(n+1) = \text{BitFTA0CarryOutput}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$.

The functor n-BitFTA0Circ(a, b_2, c_2, d_2) yielding a Boolean, strict circuit of n-BitFTA0Str(a, b_2, c_2, d_2) with denotation held in gates is defined by

(Def. 2) there exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that n-BitFTA0Str$(a, b_2, c_2, d$ $f(n)$ and $it = g(n)$ and $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1\text{GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot \text{BitFTA0Str}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$ and $g(n+1) = A +\cdot \text{BitFTA0Circ}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$ and $h(n+1) = \text{BitFTA0CarryOutput}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$.

The functor n-BitFTA0CarryOutput(a, b_2, c_2, d_2) yielding an element of InnerVertices(n-BitFTA0Str(a, b_2, c_2, d_2)) is defined by

(Def. 3) there exists a many sorted set h indexed by $\mathbb{N}$ such that $it = h(n)$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n, $h(n+1) = \text{BitFTA0CarryOutput}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), h(n))$.

Now we state the propositions:

(1) Let us consider finite sequences a, b_2, c_2, d_2, and many sorted sets f, g, h indexed by $\mathbb{N}$. Suppose $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1\text{GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot \text{BitFTA0Str}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$ and $g(n+1) = A +\cdot \text{BitFTA0Circ}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1),$

z) and $h(n+1) = \text{BitFTA0CarryOutput}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), z)$. Let us consider a natural number n. Then

(i) n-BitFTA0Str$(a, b_2, c_2, d_2) = f(n)$, and

(ii) n-BitFTA0Circ$(a, b_2, c_2, d_2) = g(n)$, and

(iii) n-BitFTA0CarryOutput$(a, b_2, c_2, d_2) = h(n)$.

PROOF: Set $f_0 = \text{1GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$. Set $g_0 = \text{1GateCircuit}(\varepsilon, \mathit{Bool}$ $\mathit{false})$. Set $h_0 = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$. Define $\mathcal{S}$(non empty many sorted signature, set, natural number) = $\$_1 +\cdot \text{BitFTA0Str}(a(\$_3 + 1), b_2(\$_3 + 1), c_2(\$_3+1), d_2(\$_3+1), \$_2)$. Define $\mathcal{A}$(non empty many sorted signature, non-empty alge over $\$_1$, set, natural number) = $\$_2 +\cdot \text{BitFTA0Circ}(a(\$_4 + 1), b_2(\$_4 + 1), c_2(\$_4+1), d_2(\$_4+1), \$_3)$. Define $\mathcal{O}$(set, natural number) = BitFTA0CarryOutput$(a(\$$ $1), b_2(\$_2 + 1), c_2(\$_2 + 1), d_2(\$_2 + 1), \$_1)$. Define $\mathcal{F}$(natural number, set) = $\text{BitFTA0CarryOutput}(a(\$_1+1), b_2(\$_1+1), c_2(\$_1+1), d_2(\$_1+1), \$_2)$. Consider f_1, g_1, h_1 being many sorted sets indexed by $\mathbb{N}$ such that n-BitFTA0Str$(a, b_2, c_2,$ $f_1(n)$ and n-BitFTA0Circ$(a, b_2, c_2, d_2) = g_1(n)$ and $f_1(0) = f_0$ and $g_1(0) = g_0$ and $h_1(0) = h_0$ and for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f_1(n)$ and $A = g_1(n)$ and $z = h_1(n)$ holds $f_1(n+1) = \mathcal{S}(S, z, n)$ and $g_1(n+1) = \mathcal{A}(S, A, z, n)$ and $h_1(n+1) = \mathcal{O}(z, n)$. $f = f_1$ and $g = g_1$ and $h = h_1$ from [4, Sch. 14]. For every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = h(n)$ holds $f(n+1) = \mathcal{S}(S, z, n)$ and $h(n+1) = \mathcal{O}(z, n)$ from [4, Sch. 15]. For every natural number n and for every set z such that $z = h(n)$ holds $h(n+1) = \mathcal{O}(z, n)$ from [4, Sch. 3]. Consider h_1 being a many sorted set indexed by $\mathbb{N}$ such that n-BitFTA0CarryOutput$(a, b_2, c_2, d_2) = h_1(n)$ and $h_1(0) = h_0$ and for every natural number n, $h_1(n+1) = \text{BitFTA0CarryOutput}(a(n+1), b_2(n+1), c_2(n+1), d_2(n+1), h_1(n))$. $h = h_1$ from [2, Sch. 15]. □

(2) Let us consider finite sequences a, b_2, c_2, d_2. Then

(i) 0-BitFTA0Str$(a, b_2, c_2, d_2) = \text{1GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$, and

(ii) 0-BitFTA0Circ$(a, b_2, c_2, d_2) = \text{1GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$, and

(iii) 0-BitFTA0CarryOutput$(a, b_2, c_2, d_2) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$.

(3) Let us consider a natural number n, and finite sequences a, b_2, c_2, d_2. Then

(i) $n+1$-BitFTA0Str$(a, b_2, c_2, d_2) = (n\text{-BitFTA0Str}(a, b_2, c_2, d_2))+\cdot\text{BitFTA0Str}(a($ $1), b_2(n+1), c_2(n+1), d_2(n+1), n$-BitFTA0CarryOutput$(a, b_2, c_2, d_2))$,

and

(ii) $n+1$-BitFTA0Circ(a, b_2, c_2, d_2) = (n-BitFTA0Circ(a, b_2, c_2, d_2)) +· BitFTA0Circ($a(n+1)$, $b_2(n+1)$, $c_2(n+1)$, $d_2(n+1)$, n-BitFTA0CarryOutput(a and

(iii) $n+1$-BitFTA0CarryOutput(a, b_2, c_2, d_2) = BitFTA0CarryOutput($a(n+1)$, $b_2(n+1)$, $c_2(n+1)$, $d_2(n+1)$, n-BitFTA0CarryOutput(a, b_2, c_2, d_2)).

The theorem is a consequence of (1).

(4) Let us consider non pair sets a, b_2, c_2, sets d_2, c_1, a state s of BitFTA0Circ(a, b_2, c_2, d_2, c_1), and a natural number k. Suppose $k \geqslant 2$. Then (Following(s, k))(BitFT b_2, c_2, d_2, c_1)) = (Following$(s, 2)$)(BitFTA0CarryOutput(a, b_2, c_2, d_2, c_1)).

Let n be a natural number and a, b_2, c_2 be finite sequences. The functor n-BitFTA0Layer1Str$(a, b_2, c_2$ yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by

(Def. 4) there exists a many sorted set f indexed by $\mathbb{N}$ such that $it = f(n)$ and $f(0) = 1$GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and for every natural number k and for every non empty many sorted signature S such that $S = f(k)$ holds $f(k+1) = S +\cdot$ BitGFA0Str$(a(k+1), b_2(k+1), c_2(k+1))$.

The functor n-BitFTA0Layer1Circ(a, b_2, c_2) yielding a Boolean, strict circuit of n-BitFTA0Layer1Str$(a, b_2, c_2$ with denotation held in gates is defined by

(Def. 5) there exist many sorted sets f, g indexed by $\mathbb{N}$ such that n-BitFTA0Layer1Str$(a, b_2$, $f(n)$ and $it = g(n)$ and $f(0) = 1$GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1$GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and for every natural number k and for every non empty many sorted signature S and for every non-empty algebra A over S such that $S = f(k)$ and $A = g(k)$ holds $f(k+1) = S +\cdot$ BitGFA0Str$(a(k+1), b_2(k+1), c_2(k+1))$ and $g(k+1) = A +\cdot$ BitGFA0Circ$(a(k+1), b_2(k+1), c_2(k+1))$.

Let a, b_2, c_2, d_2 be finite sequences. The functor n-BitFTA0Layer2Str(a, b_2, c_2, d_2) yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by

(Def. 6) there exist many sorted sets g, h indexed by $\mathbb{N}$ such that $it = g(n)$ and $g(0) = 1$GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number k and for every non empty many sorted signature S and for every set z such that $S = g(k)$ and $z = h(k)$ holds $g(k+1) = S +\cdot$ BitGFA0Str(GFA0AdderOutput$(a(k+1), b_2(k+1), c_2(k+1))$, z, $d_2(k+1)$) and $h(k+1) = $ GFA0CarryOutput$(a(k+1), b_2(k+1), c_2(k+1))$.

The functor n-BitFTA0Layer2Circ(a, b_2, c_2, d_2) yielding a Boolean, strict circuit of n-BitFTA0Layer2Str(a, b_2, c_2, d_2) with denotation held in gates is defined by

(Def. 7) there exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that n-BitFTA0Layer2Str$(a$, $f(n)$ and $it = g(n)$ and $f(0) = 1$GateCircStr$(\varepsilon, Boolean^0 \longmapsto false)$ and $g(0) = 1$GateCircuit$(\varepsilon, Boolean^0 \longmapsto false)$ and $h(0) = \langle \varepsilon, Boolean^0 \longmapsto false \rangle$ and for every natural number k and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(k)$ and $A = g(k)$ and $z = h(k)$ holds $f(k+1) = S +\cdot$ BitGFA0Str(GFA0AdderOutput$(a(k+1), b_2(k+1), c_2(k+1)), z, d_2(k+1))$ and $g(k+1) = A +\cdot$ BitGFA0Circ(GFA0AdderOutput$(a(k+1), b_2(k+1), c_2(k+1)), z, d_2(k+1))$ and $h(k+1) =$ GFA0CarryOutput$(a(k+1), b_2(k+1), c_2(k+1))$.

Let us consider a natural number n and finite sequences a, b_2, c_2, d_2. Now we state the propositions:

(5) n-BitFTA0Str$(a, b_2, c_2, d_2) = (n$-BitFTA0Layer1Str$(a, b_2, c_2) +\cdot (n$-BitFTA0Layer
PROOF: Define $\mathcal{R}$[natural number] $\equiv$ for every finite sequences a, b_2, c_2, d_2, \$1-BitFTA0Str$(a, b_2, c_2, d_2) =$ (\$1-BitFTA0Layer1Str$(a, b_2, c_2) +\cdot$ (\$1-BitFTA0Layer2Str$(a, b_2, c_2, d_2))$. $\mathcal{R}[0]$ by [9, (4)], (2). For every natural number k such that $\mathcal{R}[k]$ holds $\mathcal{R}[k+1]$ by (3), [9, (6)], [3, (23)]. For every natural number n, $\mathcal{R}[n]$ from [2, Sch. 2]. □

(6) n-BitFTA0Circ$(a, b_2, c_2, d_2) = (n$-BitFTA0Layer1Circ$(a, b_2, c_2)) +\cdot (n$-BitFTA0La
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ \$1-BitFTA0Circ$(a, b_2, c_2, d_2) =$ (\$1-BitFTA0L
(\$1-BitFTA0Layer2Circ$(a, b_2, c_2, d_2))$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (3), [9, (59), (48)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

Now we state the propositions:

(7) Let us consider a natural number n, and nonpair yielding finite sequences a, b_2, c_2, d_2. Then InputVertices(n-BitFTA0Layer1Str(a, b_2, c_2) misses InnerVertices(n-BitFTA0Layer2Str(a, b_2, c_2, d_2)).

(8) Let us consider a natural number n, nonpair yielding finite sequences a, b_2, c_2, and a state s of n-BitFTA0Layer1Circ(a, b_2, c_2). Then Following$(s, 2)$ is stable.
PROOF: Set $A_0 = 1$GateCircuit$(\varepsilon, Boolean^0 \longmapsto false)$. Define $\mathcal{P}$[natural number] $\equiv$ for every state s of \$1-BitFTA0Layer1Circ(a, b_2, c_2), Following$(s, 2)$ is stable. 0-BitFTA0Layer1Circ$(a, b_2, c_2) = A_0$. $\mathcal{P}[0]$ by [3, (14)], [4, (2), (4)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [9, (60)], [15, (32)], [3, (5)], [15, (35), (40)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(9) Let us consider an unsplit, non void, non empty many sorted signature S with arity held in gates and Boolean denotation held in gates, and a Boolean, non-empty circuit A of S with denotation held in gates. Suppose $\langle \varepsilon,\ \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle \in \mathrm{InnerVertices}(S)$. Let us consider a state s of A. Then $(\mathrm{Following}(s))(\langle \varepsilon,\ \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle) = \mathit{false}$.

(10) Let us consider a natural number n, nonpair yielding finite sequences a, b_2, c_2, d_2, and a state s of n-BitFTA0Layer2Circ(a, b_2, c_2, d_2). Suppose $s(\langle \varepsilon,\ \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle) = \mathit{false}$. Then Following$(s, 2)$ is stable.
Proof: Set $A_0 = 1$GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$. Define $\mathcal{P}$[natural number] $\equiv$ for every state s of $\$_1$-BitFTA0Layer2Circ$(a, b_2, c_2, d_2)$ such that $s(\langle \varepsilon,\ \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle) = \mathit{false}$ holds Following$(s, 2)$ is stable. 0-BitFTA0Layer2Circ$(a, b_2, c_2, d_2) = A_0$. $\mathcal{P}[0]$ by [3, (14)], [4, (2), (4)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [9, (60), (26)], [7, (49)], [2, (6)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(11) Let us consider a natural number n, nonpair yielding finite sequences a, b_2, c_2, d_2 with length n, and a state s of n-BitFTA0Circ(a, b_2, c_2, d_2). Then Following$(s, 4)$ is stable.
Proof: Set $S_1 = n$-BitFTA0Layer1Str$(a, b_2, c_2$. Set $S_2 = n$-BitFTA0Layer2Str$(a, b_2$ Set $C_1 = n$-BitFTA0Layer1Circ(a, b_2, c_2). Set $C_2 = n$-BitFTA0Layer2Circ$(a, b_2, c_2,$ Set $S = n$-BitFTA0Str(a, b_2, c_2, d_2). Set $C = n$-BitFTA0Circ(a, b_2, c_2, d_2). $S = S_1 +\cdot S_2$. $C = C_1 +\cdot C_2$. Reconsider $t = s$ as a state of $C_1 +\cdot C_2$. Reconsider $s_1 = t\restriction$(the carrier of S_1) as a state of C_1. Reconsider $s_2 = \mathrm{Following}(t, 2)\restriction$(the carrier of S_2) as a state of C_2. InputVertices(S_1) misses InnerVertices(S_2). Following$(s_1, 2)$ is stable. Following$(s_2, 2)$ is stable by [9, (47), (11)], (9), [7, (49)]. □

2. Stability of a Parallel 4-2 Compressor Circuit (TYPE-1)

Let n be a natural number and a, b_1, c_2, d_1 be finite sequences. The functor n-BitFTA1Str(a, b_1, c_2, d_1) yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by

(Def. 8) there exist many sorted sets f, h indexed by $\mathbb{N}$ such that $\mathit{it} = f(n)$ and $f(0) = 1$GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon,\ \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle$ and for every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitFTA1Str$(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$ and $h(n+1) =$ BitFTA1CarryOutput$(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$.

The functor n-BitFTA1Circ(a, b_1, c_2, d_1) yielding a Boolean, strict circuit of n-BitFTA1Str(a, b_1, c_2, d_1) with denotation held in gates is defined by

(Def. 9) there exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that n-BitFTA1Str$(a, b_1, c_2, d$ $f(n)$ and $it = g(n)$ and $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1\text{GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot \text{BitFTA1Str}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$ and $g(n+1) = A +\cdot \text{BitFTA1Circ}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$ and $h(n+1) = \text{BitFTA1CarryOutput}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$.

The functor n-BitFTA1CarryOutput(a, b_1, c_2, d_1) yielding an element of InnerVertices(n-BitFTA1Str(a, b_1, c_2, d_1)) is defined by

(Def. 10) there exists a many sorted set h indexed by $\mathbb{N}$ such that $it = h(n)$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n, $h(n+1) = \text{BitFTA1CarryOutput}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), h(n))$.

Now we state the propositions:

(12) Let us consider finite sequences a, b_1, c_2, d_1, and many sorted sets f, g, h indexed by $\mathbb{N}$. Suppose $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1\text{GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot \text{BitFTA1Str}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$ and $g(n+1) = A +\cdot \text{BitFTA1Circ}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$ and $h(n+1) = \text{BitFTA1CarryOutput}(a(n+1), b_1(n+1), c_2(n+1), d_1(n+1), z)$. Let us consider a natural number n. Then

(i) n-BitFTA1Str$(a, b_1, c_2, d_1) = f(n)$, and

(ii) n-BitFTA1Circ$(a, b_1, c_2, d_1) = g(n)$, and

(iii) n-BitFTA1CarryOutput$(a, b_1, c_2, d_1) = h(n)$.

PROOF: Set $f_0 = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$. Set $g_0 = 1\text{GateCircuit}(\varepsilon, \mathit{Bool}$ $\mathit{false})$. Set $h_0 = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$. Define $\mathcal{S}$(non empty many sorted signature, set, natural number) = $\$_1 +\cdot \text{BitFTA1Str}(a(\$_3+1), b_1(\$_3+1), c_2(\$_3+1), d_1(\$_3+1), \$_2)$. Define $\mathcal{A}$(non empty many sorted signature, non-empty alge over $\$_1$, set, natural number) = $\$_2 +\cdot \text{BitFTA1Circ}(a(\$_4+1), b_1(\$_4+1), c_2(\$_4+1), d_1(\$_4+1), \$_3)$. Define $\mathcal{O}$(set, natural number) = BitFTA1CarryOutput$(a(\$$ $1), b_1(\$_2+1), c_2(\$_2+1), d_1(\$_2+1), \$_1)$. Define $\mathcal{F}$(natural number, set) =

BitFTA1CarryOutput(a($\$_1$+1), b_1($\$_1$+1), c_2($\$_1$+1), d_1($\$_1$+1), $\$_2$). Consider f_1, g_1, h_1 being many sorted sets indexed by $\mathbb{N}$ such that n-BitFTA1Str(a, b_1, c_2, $f_1(n)$ and n-BitFTA1Circ$(a, b_1, c_2, d_1) = g_1(n)$ and $f_1(0) = f_0$ and $g_1(0) = g_0$ and $h_1(0) = h_0$ and for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f_1(n)$ and $A = g_1(n)$ and $z = h_1(n)$ holds $f_1(n+1) = \mathcal{S}(S, z, n)$ and $g_1(n+1) = \mathcal{A}(S, A, z, n)$ and $h_1(n+1) = \mathcal{O}(z, n)$. $f = f_1$ and $g = g_1$ and $h = h_1$ from [4, Sch. 14]. For every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = h(n)$ holds $f(n + 1) = \mathcal{S}(S, z, n)$ and $h(n + 1) = \mathcal{O}(z, n)$ from [4, Sch. 15]. For every natural number n and for every set z such that $z = h(n)$ holds $h(n + 1) = \mathcal{O}(z, n)$ from [4, Sch. 3]. Consider h_1 being a many sorted set indexed by $\mathbb{N}$ such that n-BitFTA1CarryOutput$(a, b_1, c_2, d_1) = h_1(n)$ and $h_1(0) = h_0$ and for every natural number n, $h_1(n+1)$ = BitFTA1CarryOutput($a(n+1)$, $b_1(n+1)$, $c_2(n + 1)$, $d_1(n + 1)$, $h_1(n)$). $h = h_1$ from [2, Sch. 15]. □

(13) Let us consider finite sequences a, b_1, c_2, d_1. Then

(i) 0-BitFTA1Str(a, b_1, c_2, d_1) = 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$, and

(ii) 0-BitFTA1Circ(a, b_1, c_2, d_1) = 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$, and

(iii) 0-BitFTA1CarryOutput$(a, b_1, c_2, d_1) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$.

(14) Let us consider a natural number n, and finite sequences a, b_1, c_2, d_1. Then

(i) n+1-BitFTA1Str(a, b_1, c_2, d_1) = (n-BitFTA1Str(a, b_1, c_2, d_1))+·BitFTA1Str($a($ 1), $b_1(n+1)$, $c_2(n+1)$, $d_1(n+1)$, n-BitFTA1CarryOutput(a, b_1, c_2, d_1)), and

(ii) $n + 1$-BitFTA1Circ(a, b_1, c_2, d_1) = (n-BitFTA1Circ(a, b_1, c_2, d_1)) +· BitFTA1Circ($a(n+1)$, $b_1(n+1)$, $c_2(n+1)$, $d_1(n+1)$, n-BitFTA1CarryOutput(a and

(iii) n+1-BitFTA1CarryOutput(a, b_1, c_2, d_1) = BitFTA1CarryOutput($a(n+$ 1), $b_1(n+1)$, $c_2(n+1)$, $d_1(n+1)$, n-BitFTA1CarryOutput(a, b_1, c_2, d_1)).

The theorem is a consequence of (12).

(15) Let us consider non pair sets a, b_1, c_2, sets d_1, c_1, a state s of BitFTA1Circ(a, b_1, c_2, d_1, c_1), and a natural number k. Suppose $k \geqslant 2$. Then (Following(s, k))(BitFT b_1, c_2, d_1, c_1)) = (Following$(s, 2)$)(BitFTA1CarryOutput(a, b_1, c_2, d_1, c_1)).

Let n be a natural number and a, b_1, c_2 be finite sequences. The functor n-BitFTA1Layer1Str(a, b_1, c_2) yielding an unsplit, non void, strict, non empty

many sorted signature with arity held in gates and Boolean denotation held in gates is defined by

(Def. 11) there exists a many sorted set f indexed by $\mathbb{N}$ such that $it = f(n)$ and $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and for every natural number k and for every non empty many sorted signature S such that $S = f(k)$ holds $f(k+1) = S +\cdot \text{BitGFA1Str}(a(k+1), b_1(k+1), c_2(k+1))$.

The functor $n\text{-BitFTA1Layer1Circ}(a, b_1, c_2)$ yielding a Boolean, strict circuit of $n\text{-BitFTA1Layer1Str}(a, b_1, c_2)$ with denotation held in gates is defined by

(Def. 12) there exist many sorted sets f, g indexed by $\mathbb{N}$ such that $n\text{-BitFTA1Layer1Str}(a, b_1, f(n)$ and $it = g(n)$ and $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1\text{GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and for every natural number k and for every non empty many sorted signature S and for every non-empty algebra A over S such that $S = f(k)$ and $A = g(k)$ holds $f(k+1) = S +\cdot \text{BitGFA1Str}(a(k+1), b_1(k+1), c_2(k+1))$ and $g(k+1) = A +\cdot \text{BitGFA1Circ}(a(k+1), b_1(k+1), c_2(k+1))$.

Let a, b_1, c_2, d_1 be finite sequences. The functor $n\text{-BitFTA1Layer2Str}(a, b_1, c_2, d_1)$ yielding an unsplit, non void, strict, non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by

(Def. 13) there exist many sorted sets g, h indexed by $\mathbb{N}$ such that $it = g(n)$ and $g(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number k and for every non empty many sorted signature S and for every set z such that $S = g(k)$ and $z = h(k)$ holds $g(k+1) = S +\cdot \text{BitGFA2Str}(\text{GFA1AdderOutput}(a(k+1), b_1(k+1), c_2(k+1)), z, d_1(k+1))$ and $h(k+1) = \text{GFA1CarryOutput}(a(k+1), b_1(k+1), c_2(k+1))$.

The functor $n\text{-BitFTA1Layer2Circ}(a, b_1, c_2, d_1)$ yielding a Boolean, strict circuit of $n\text{-BitFTA1Layer2Str}(a, b_1, c_2, d_1)$ with denotation held in gates is defined by

(Def. 14) there exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that $n\text{-BitFTA1Layer2Str}(a, f(n)$ and $it = g(n)$ and $f(0) = 1\text{GateCircStr}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $g(0) = 1\text{GateCircuit}(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$ and for every natural number k and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(k)$ and $A = g(k)$ and $z = h(k)$ holds $f(k+1) = S +\cdot \text{BitGFA2Str}(\text{GFA1AdderOutput}(a(k+1), b_1(k+1), c_2(k+1)), z, d_1(k+1))$ and $g(k+1) = A +\cdot \text{BitGFA2Circ}(\text{GFA1AdderOutput}(a(k+1), b_1(k+1), c_2(k+1)), z, d_1(k+1))$ and $h(k+1) = \text{GFA1CarryOutput}(a(k+1), b_1(k+$

$1), c_2(k+1))$.

Let us consider a natural number n and finite sequences a, b_1, c_2, d_1. Now we state the propositions:

(16) n-BitFTA1Str$(a, b_1, c_2, d_1) = (n$-BitFTA1Layer1Str$(a, b_1, c_2))+\cdot(n$-BitFTA1Layer
PROOF: Define $\mathcal{R}$[natural number] $\equiv$ for every finite sequences a, b_1, c_2, d_1, $\$_1$-BitFTA1Str$(a, b_1, c_2, d_1) = (\$_1$-BitFTA1Layer1Str$(a, b_1, c_2)) +\cdot$ ($\$_1$-BitFTA1Layer2Str$(a, b_1, c_2, d_1)$). $\mathcal{R}[0]$ by [9, (4)], (13). For every natural number k such that $\mathcal{R}[k]$ holds $\mathcal{R}[k+1]$ by (14), [9, (6)], [3, (23)]. For every natural number n, $\mathcal{R}[n]$ from [2, Sch. 2]. □

(17) n-BitFTA1Circ$(a, b_1, c_2, d_1) = (n$-BitFTA1Layer1Circ$(a, b_1, c_2))+\cdot(n$-BitFTA1La
PROOF: Define $\mathcal{P}$[natural number] $\equiv \$_1$-BitFTA1Circ$(a, b_1, c_2, d_1) = (\$_1$-BitFTA1L
($\$_1$-BitFTA1Layer2Circ$(a, b_1, c_2, d_1)$). $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (14), [9, (59), (48)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

Now we state the propositions:

(18) Let us consider a natural number n, and nonpair yielding finite sequences a, b_1, c_2, d_1. Then InputVertices(n-BitFTA1Layer1Str(a, b_1, c_2)) misses InnerVertices(n-BitFTA1Layer2Str(a, b_1, c_2, d_1)).

(19) Let us consider a natural number n, nonpair yielding finite sequences a, b_1, c_2, and a state s of n-BitFTA1Layer1Circ(a, b_1, c_2). Then Following$(s, 2)$ is stable.
PROOF: Set $A_0 =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$. Define $\mathcal{P}$[natural number] $\equiv$ for every state s of $\$_1$-BitFTA1Layer1Circ$(a, b_1, c_2)$, Following$(s, 2)$ is stable. 0-BitFTA1Layer1Circ$(a, b_1, c_2) = A_0$. $\mathcal{P}[0]$ by [3, (14)], [4, (2), (4)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [9, (60)], [15, (64)], [3, (5)], [15, (67), (72)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(20) Let us consider a natural number n, nonpair yielding finite sequences a, b_1, c_2, d_1, and a state s of n-BitFTA1Layer2Circ(a, b_1, c_2, d_1). Suppose $s(\langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle) = \mathit{false}$. Then Following$(s, 2)$ is stable.
PROOF: Set $A_0 =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$. Define $\mathcal{P}$[natural number] $\equiv$ for every state s of $\$_1$-BitFTA1Layer2Circ$(a, b_1, c_2, d_1)$ such that $s(\langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle) = \mathit{false}$ holds Following$(s, 2)$ is stable. 0-BitFTA1Layer2Circ$(a, b_1, c_2, d_1) = A_0$. $\mathcal{P}[0]$ by [3, (14)], [4, (2), (4)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [9, (60), (26)], [7, (49)], [2, (6)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(21) Let us consider a natural number n, nonpair yielding finite sequences a, b_1, c_2, d_1 with length n, and a state s of n-BitFTA1Circ(a, b_1, c_2, d_1). Then Following$(s, 4)$ is stable.

PROOF: Set $S_1 = n$-BitFTA1Layer1Str(a, b_1, c_2). Set $S_2 = n$-BitFTA1Layer2Str$(a, b$
Set $C_1 = n$-BitFTA1Layer1Circ(a, b_1, c_2). Set $C_2 = n$-BitFTA1Layer2Circ$(a, b_1, c_2, c$
Set $S = n$-BitFTA1Str(a, b_1, c_2, d_1). Set $C = n$-BitFTA1Circ(a, b_1, c_2, d_1).
$S = S_1 +\cdot S_2$. $C = C_1 +\cdot C_2$. Reconsider $t = s$ as a state of $C_1 +\cdot C_2$. Reconsider $s_1 = t\restriction$(the carrier of S_1) as a state of C_1. Reconsider $s_2 =$ Following$(t, 2)\restriction$(the carrier of S_2) as a state of C_2. InputVertices(S_1) misses InnerVertices(S_2). Following$(s_1, 2)$ is stable. Following$(s_2, 2)$ is stable by [9, (47), (11)], (9), [7, (49)]. □

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