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An Explicit Construction of Free Modules in Mizar

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Summary. In this paper, we present a standard method in Mizar for explicitly constructing a free left R-module from a given set of generators. For a general set X of generators, infinite linear combinations cannot be formed in a purely algebraic setting. We therefore impose the finite-support condition and realize the free left R-module on X as the module of all finitely supported functions from X to R. When X is empty, this construction yields the trivial module.

Although the module structure itself can be defined over an arbitrary ring, the results concerning bases require the ring to be non-degenerated. We formalize this construction and prove that, when X is non-empty, it satisfies the universal mapping property (UMP), a useful property for subsequent applications. Furthermore, we show that, over a non-degenerated ring, `Free_LMod(X,R)` is a free module for every set X. This formalization provides an important tool for the further development of algebra and commutative algebra in the Mizar Mathematical Library.

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Introduction

In standard textbooks, a free module is often presented as a direct sum $\bigoplus R$ with only finitely many non-zero components (see, for example, [1]). Such

presentations emphasize essential properties, while the underlying construction is left implicit.

In this paper, we treat free modules generated by an arbitrary set constructively, without assuming the existence of a basis, and derive the universal mapping property (UMP) from the construction itself. For a general set X, linear combinations of generators would involve infinite sums, which are not defined in a purely algebraic setting. The finite-support condition resolves this point: the functions from X to R with finite support form a free R-module with basis X. The construction follows [6] and a lecture note by T. Ozawa [14], in which the free module on X is realized explicitly as the module of finitely supported functions from X to R. In the Mizar Mathematical Library, a free module is defined by the existence of a basis (see [13]), whereas our approach starts from a given generating set and obtains a basis as a theorem.

As a brief remark, in Lean/mathlib, functions with finite support (`finsupp`) are used to handle linear combinations over infinite bases (see [11]).

This paper consists of five sections. The contents of each section are as follows. In Section 1 we work with an arbitrary set X and an arbitrary ring R; from Section 2 onward we assume that X is non-empty and that R is non-degenerated, and the case $X = \emptyset$ is treated separately in Section 5. All modules are assumed to be left R-modules.

Section 1. Function spaces and free modules In Section 1, we consider the left R-module consisting of all functions from X to R, and construct the submodule of finitely supported functions. We regard this submodule as a candidate for the free left R-module on X. In the next section, we introduce a standard set of generators and use the finite-support condition to show that every element can be written as a finite linear combination of these generators. This module is denoted by $\mathcal{F}_{\mathrm{LMod}}(X)$. In the Mizar code, it is defined as `Free_LMod(X,R)` (`LMODFREE:def 14`).

Section 2. Basis map and reconstruction Let X be a non-empty set, and consider the left R-module $\mathcal{F}_{\mathrm{LMod}}(X)$ consisting of finitely supported functions from X to R. We define a map

$$\iota_X : X \longrightarrow \mathcal{F}_{\mathrm{LMod}}(X),$$

which assigns the standard generator $\iota_X(x)$ to each $x \in X$. We call this map the basis map. In the Mizar code, it is written as `IotaMap(X,R)`. The notation ι follows [14]; in [6], the corresponding map is denoted by $f(s)$. For each $x \in X$,

the function $\iota_X(x)$ is defined by

$$(\iota_X(x))(y) = \begin{cases} 1_R, & y = x, \\ 0_R, & y \neq x. \end{cases}$$

For $f \in \mathcal{F}_{\mathrm{LMod}}(X)$ and $x \in X$, we define the evaluation map by

$$\mathrm{ev}(x)(f) = f(x).$$

With this notation, the following reconstruction formula holds:

$$\sum_{x \in X} (\mathrm{ev}(x)(f)) \cdot \iota_X(x) = f.$$

In other words, $\mathrm{ev}(x)(f)$ gives the coefficient of f at the standard generator $\iota_X(x)$, and the formula expresses f as a finite linear combination of the standard generators. Since f has finite support, only finitely many terms in the sum are non-zero. To handle this finite sum formally, the elements of the support of f are numbered and arranged as a finite sequence. For this purpose, `idxseq(f)` lists the elements of the support of f in a standard order determined by `canFS`. The sequence `F_idxseq(f)` lists the corresponding terms

$$(\mathrm{ev}(x)(f)) \cdot \iota_X(x),$$

with its i-th term corresponding to the i-th element of `idxseq(f)`. This common numbering allows the reconstruction theorem to be proved by induction on the length of the sequence. It follows that every element f has a representation as a finite linear combination of the standard generators. The reconstruction formula is formalized as

`f = SumF_idxseq(f)`

(`LMODFREE:16`).

Section 3. Universal mapping property Let X be a non-empty set. A pair (M, μ), where M is a left R-module and $\mu : X \to M$ is a map, is called universal (`LMODFREE:def 33`) if, for every left R-module N and every map $\nu : X \to N$, there exists a unique R-linear map $\varphi : M \to N$ such that $\varphi \circ \mu = \nu$. The free left R-module $\mathcal{F}_{\mathrm{LMod}}(X)$ constructed in Section 2, together with the basis map ι_X, has this universal mapping property. In other words, once the image of each generator $\iota_X(x)$ is given, there is a unique linear map extending this assignment. Therefore, to construct a linear map from $\mathcal{F}_{\mathrm{LMod}}(X)$ to any left R-module, it is enough to specify the images of the generators; and if two linear maps agree on all generators, they agree on the whole module. This universal mapping property follows from the reconstruction formula in Section 2 and is formalized as `LMODFREE:27`.

Section 4. Linear independence and basis In Section 4, we show that the image of the basis map ι_X is linearly independent and forms a basis of $\mathcal{F}_{\mathrm{LMod}}(X)$. Consequently, $\mathcal{F}_{\mathrm{LMod}}(X)$ is a free left R-module. Neither fact is assumed: linear independence is verified by reading off coefficients with the evaluation maps, and the spanning property follows from the reconstruction formula. This derivation is fully formalized in Mizar [4], [5].

Section 5. The free module generated by the empty set The construction in Section 1 also applies to an arbitrary set X, including the empty set. In Section 5 we show that $\mathcal{F}_{\mathrm{LMod}}(\emptyset)$ coincides, as a structure, with the trivial left R-module `TrivialLMod(R)` of [12], and that it is free: the empty subset is vacuously linearly independent and forms a basis of the trivial module. Combining this with Section 4, we obtain the general result that `Free_LMod(X,R)` is free for every set X over a non-degenerated ring; this fact is also registered as a cluster, so that the Mizar system recognizes the freeness of the term automatically.

In this way, the assignment

$$X \longmapsto \mathcal{F}_{\mathrm{LMod}}(X)$$

is defined on all sets and can be viewed as the object assignment of the free-module functor from sets to left R-modules.

1. Function Spaces and Free Modules

From now on R denotes a ring, X denotes a set, o, x denote objects, a, b denote elements of the carrier of R, M, N denote left modules over R, and i denotes a natural number.

Let us consider R, X, and N. The functor $0\_\mathrm{Funcs}(X, N)$ yielding an element of (the carrier of $N)^X$ is defined by the term

(Def. 1) $X \longmapsto 0_N$.

One can verify that (the carrier of $N)^X$ is non empty.

The functor $\mathrm{ADD}(X, N)$ yielding a binary operation on (the carrier of $N)^X$ is defined by

(Def. 2) for every elements f, g of (the carrier of $N)^X$, $it(f, g) =$ (the addition of $N)^\circ(f, g)$.

From now on f, g, h denote elements of (the carrier of $N)^X$.

Let us consider R, X, and N. Let F be a function from (the carrier of R) $\times$ (the carrier of N) into the carrier of N, a be an element of the carrier of R, and f be a function from X into N. One can check that the functor $F^\circ(a, f)$ yields an element of (the carrier of $N)^X$. The functor $\mathrm{LMULT}(X, N)$ yielding

a function from (the carrier of R) $\times$ (the carrier of N)^X into (the carrier of N)^X is defined by

(Def. 3) for every element a of the carrier of R and for every element f of (the carrier of N)^X and for every object x such that $x \in X$ holds $it(a, f)(x) =$ (the left multiplication of N)$(a, f(x))$.

The functor Func-Mod(X, N) yielding a non empty vector space structure over R is defined by the term

(Def. 4) $\langle$(the carrier of N)^X, ADD(X, N), 0_Funcs(X, N), LMULT$(X, N)\rangle$.

Module structure on function spaces and on finitely supported functions.

Let us consider R and X. The functor 0_Funcs(X, R) yielding an element of (the carrier of R)^X is defined by the term

(Def. 5) $X \longmapsto 0_R$.

The functor ADD(X, R) yielding a binary operation on (the carrier of R)^X is defined by

(Def. 6) for every elements f, g of (the carrier of R)^X, $it(f, g) =$ (the addition of R)$^\circ(f, g)$.

In the sequel f, g, h denote elements of (the carrier of R)^X.

Now we state the proposition:

(1) $h =$ (ADD(X, R))(f, g) if and only if for every object x such that $x \in X$ holds $h(x) =$ (the addition of R)$(f(x), g(x))$.

Proof: If $h =$ (ADD(X, R))(f, g), then for every object x such that $x \in X$ holds $h(x) =$ (the addition of R)$(f(x), g(x))$ by [15, (22)]. For every object x such that $x \in \operatorname{dom} h$ holds $h(x) =$ (ADD(X, R))$(f, g)(x)$ by [15, (22)]. □

Let us consider R and X. Let F be a function from (the carrier of R) $\times$ (the carrier of R) into the carrier of R, a be an element of the carrier of R, and f be a function from X into R. Let us note that the functor $F^\circ(a, f)$ yields an element of (the carrier of R)^X. The functor LMULT(X, R) yielding a function from (the carrier of R) $\times$ (the carrier of R)^X into (the carrier of R)^X is defined by

(Def. 7) for every element a of the carrier of R and for every element f of (the carrier of R)^X and for every object x such that $x \in X$ holds $it(a, f)(x) =$ (the multiplication of R)$(a, f(x))$.

Now we state the proposition:

(2) Let us consider an element a of the carrier of R, and elements f, h of (the carrier of R)^X. Then $h =$ (LMULT(X, R))(a, f) if and only if for every x such that $x \in X$ holds $h(x) =$ (the multiplication of R)$(a, f(x))$.

PROOF: If for every x such that $x \in X$ holds $h(x) =$ (the multiplication of R)$(a, f(x))$, then $h =$ (LMULT(X, R))(a, f) by [9, (92)]. □

Let us consider R and X. The functor fs-Funcs(X, R) yielding a subset of (the carrier of $R)^X$ is defined by the term

(Def. 8) $\{x$, where x is a function from X into the carrier of $R : x$ is finite-Support$\}$.

Let us note that fs-Funcs(X, R) is non empty and there exists a function from X into the carrier of R which is finite-Support.

Now we state the propositions:

(3) Let us consider finite-Support functions f, g from X into the carrier of R. Then (ADD(X, R))$(\langle f, g \rangle)$ is a finite-Support function from X into the carrier of R.

PROOF: Reconsider $f_1 = f$, $g_1 = g$ as an element of (the carrier of $R)^X$. Reconsider $h =$ (ADD(X, R))(f_1, g_1) as an element of (the carrier of $R)^X$. For every object o such that $o \in$ Support h holds $o \in$ Support $f_1 \cup$ Support g_1 by [15, (22)]. □

(4) Let us consider an element a of the carrier of R, and a finite-Support function g from X into the carrier of R. Then (LMULT(X, R))(a, g) is a finite-Support function from X into the carrier of R.

PROOF: Reconsider $F =$ (LMULT(X, R))(a, g) as an element of (the carrier of $R)^X$. For every object o such that $o \in$ Support F holds $o \in$ Support g by [9, (5)], (2). □

From now on f, g, h denote elements of fs-Funcs(X, R).

Let X be a non empty set and Y be a non empty additive loop structure. The functor FuncAdd(X, Y) yielding a binary operation on (the carrier of $Y)^X$ is defined by

(Def. 9) for every elements f, g of (the carrier of $Y)^X$, $it(f, g) =$ (the addition of $Y)^\circ(f, g)$.

Let us consider R and X. The functor fs-ADD(X, R) yielding a binary operation on fs-Funcs(X, R) is defined by the term

(Def. 10) ADD$(X, R)\restriction$(fs-Funcs$(X, R) \times$ fs-Funcs$(X, R))$.

The functor fs-LMULT(X, R) yielding a function from (the carrier of $R) \times$ fs-Funcs(X, R) into fs-Funcs(X, R) is defined by the term

(Def. 11) LMULT$(X, R)\restriction$((the carrier of $R) \times$ fs-Funcs$(X, R))$.

The functor 0-fs-Zero(X, R) yielding an element of fs-Funcs(X, R) is defined by the term

(Def. 12) 0_Funcs(X, R).

Let us consider R and X. The functor FS-Funcs(X, R) yielding a non empty vector space structure over R is defined by the term

(Def. 13) $\langle \text{fs-Funcs}(X, R), \text{fs-ADD}(X, R), \text{0-fs-Zero}(X, R), \text{fs-LMULT}(X, R) \rangle$.

Now we state the propositions:

(5) Let us consider elements v_1, w_1 of (the carrier of $R)^X$, and elements v, w of FS-Funcs(X, R). If $v_1 = v$ and $w_1 = w$, then $v+w = (\text{ADD}(X, R))(v_1, w_1)$.

(6) FS-Funcs(X, R) is a left module over R.

PROOF: FS-Funcs(X, R) is Abelian by [8, (47)]. FS-Funcs(X, R) is add-associative. FS-Funcs(X, R) is right zeroed. FS-Funcs(X, R) is right complementable. FS-Funcs(X, R) is vector distributive. FS-Funcs(X, R) is scalar distributive. FS-Funcs(X, R) is scalar associative. FS-Funcs(X, R) is scalar unital. □

Let us consider R and X. One can verify that FS-Funcs(X, R) is Abelian, add-associative, right zeroed, left zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

Let us consider R and X. The functor Free-LMod(X, R) yielding a non empty vector space structure over R is defined by the term

(Def. 14) FS-Funcs(X, R).

Let us observe that Free-LMod(X, R) is Abelian, add-associative, right zeroed, left zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

2. BASIS MAP AND RECONSTRUCTION

In the sequel X denotes a non empty set, R denotes a non degenerated ring, and M, N denote left modules over R.

Let us consider R and X. Let f be an element of Free-LMod(X, R). The functor R2P(f) yielding an element of fs-Funcs(X, R) is defined by the term

(Def. 15) f.

The functor ${}^{@}f$ yielding an element of (the carrier of $R)^X$ is defined by the term

(Def. 16) f.

Let us consider X and R. Let a be an element of X. The functor iota(a, R) yielding a function from X into Ω_R is defined by

(Def. 17) for every object x such that $x \in X$ holds if $x = a$, then $it(x) = 1_R$ and if $x \neq a$, then $it(x) = 0_R$.

Let us note that iota(a, R) is (the carrier of R)-valued and iota(a, R) is finite-Support.

Let x be an element of X. The functor Iota(x, R) yielding an element of Free-LMod(X, R) is defined by the term

(Def. 18) iota(x, R).

Let us consider R and X. The functor IotaMap(X, R) yielding a function from X into Free-LMod(X, R) is defined by

(Def. 19) for every element x of X, $it(x) = \mathrm{Iota}(x, R)$.

One can verify that IotaMap(X, R) is one-to-one.

Let x be an element of X. The functor ev(x, R) yielding a function from Free-LMod(X, R) into R is defined by

(Def. 20) for every element f of Free-LMod(X, R), $it(f) = ({}^{@}f)(x)$.

Let f be an element of Free-LMod(X, R). The functor ev-iota(f) yielding a function from X into Free-LMod(X, R) is defined by

(Def. 21) for every element x of X, $it(x) = (\mathrm{ev}(x, R))(f) \cdot (\mathrm{IotaMap}(X, R))(x)$.

Observe that Support ${}^{@}f$ is finite as a set and $0_{\text{Free-LMod}(X,R)}$ is zero and there exists an element of Free-LMod(X, R) which is non zero.

Let f be an element of Free-LMod(X, R). The functor idxseq(f) yielding a finite sequence of elements of X is defined by the term

(Def. 22) CFS(Support ${}^{@}f$).

Now we state the proposition:

(7) Let us consider an element f of Free-LMod(X, R). Then Support ev-iota(f) = Support ${}^{@}f$.

PROOF: Support ev-iota$(f) \subseteq$ Support ${}^{@}f$ by [10, (14)]. □

Let us consider X and R. Let s be a finite sequence of elements of the carrier of Free-LMod(X, R) and x be an element of X. The functor ev(s, x) yielding a finite sequence of elements of the carrier of R is defined by

(Def. 23) $\mathrm{dom}\, it = \mathrm{dom}\, s$ and for every natural number i such that $i \in \mathrm{dom}\, s$ holds $it(i) = (\mathrm{ev}(x, R))(s_{/i})$.

Now we state the propositions:

(8) Let us consider an element x of X, an element f of Free-LMod(X, R), and an element a of R. Then $({}^{@}a \cdot f)(x) = a \cdot ({}^{@}f)(x)$. The theorem is a consequence of (2).

(9) Let us consider an element x of X, and elements f, g of Free-LMod(X, R). Then

(i) $({}^{@}(f + g))(x) = ({}^{@}f)(x) + ({}^{@}g)(x)$, and

(ii) $(\mathrm{ev}(x, R))(f + g) = (\mathrm{ev}(x, R))(f) + (\mathrm{ev}(x, R))(g)$.

The theorem is a consequence of (5) and (1).

(10) Let us consider an element x of X. Then $(\mathrm{ev}(x,R))(0_{\text{Free-LMod}(X,R)}) = 0_R$.

(11) Let us consider a natural number N_0, a finite sequence F of elements of the carrier of Free-LMod(X,R), and an element x of X. Suppose $\operatorname{len} F = N_0 + 1$. Then $\mathrm{ev}(F,x) = \mathrm{ev}(F{\restriction}N_0,x) \frown \langle(\mathrm{ev}(x,R))(F_{/\operatorname{len} F})\rangle$.

PROOF: For every natural number k such that $1 \leqslant k \leqslant \operatorname{len}(\mathrm{ev}(F,x))$ holds $(\mathrm{ev}(F,x))(k) = (\mathrm{ev}(F{\restriction}N_0,x) \frown \langle(\mathrm{ev}(x,R))(F_{/\operatorname{len} F})\rangle)(k)$ by [3, (9)], [16, (18)], [8, (47)], [3, (6), (4)]. □

(12) Let us consider a finite sequence F of elements of the carrier of Free-LMod(X,R), and an element x of X. Then $(\mathrm{ev}(x,R))(\sum F) = \sum \mathrm{ev}(F,x)$. The theorem is a consequence of (10), (9), and (11).

Let us consider X and R. Let f be an element of Free-LMod(X,R). The functor F-idxseq(f) yielding a finite sequence of elements of the carrier of Free-LMod(X,R) is defined by

(Def. 24) $\operatorname{len} it = \operatorname{len}(\mathrm{idxseq}(f))$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = (\mathrm{ev\text{-}iota}(f))((\mathrm{idxseq}(f))(i))$.

Let us consider an element f of Free-LMod(X,R) and an element x of X. Now we state the propositions:

(13) If $x \notin \operatorname{Support}{}^{@}f$, then $0_R = ({}^{@}\sum \text{F-idxseq}(f))(x)$.

PROOF: For every natural number i such that $i \in \operatorname{dom}(\mathrm{ev}(\text{F-idxseq}(f),x))$ holds $(\mathrm{ev}(\text{F-idxseq}(f),x))(i) = 0_R$ by [8, (3), (49)]. □

(14) Suppose $x \in \operatorname{Support}{}^{@}f$. Then there exists a natural number k such that

(i) $x = (\mathrm{CFS}(\operatorname{Support}{}^{@}f))(k)$, and

(ii) $k \in \operatorname{dom}(\mathrm{CFS}(\operatorname{Support}{}^{@}f))$, and

(iii) $(\mathrm{ev}(\text{F-idxseq}(f),x))(k) = ({}^{@}f)(x)$.

PROOF: Consider o being an object such that $o \in \operatorname{dom}(\mathrm{CFS}(\operatorname{Support}{}^{@}f))$ and $x = (\mathrm{CFS}(\operatorname{Support}{}^{@}f))(o)$. Reconsider $n_1 = o$ as a natural number. $(\mathrm{ev}(\text{F-idxseq}(f),x))(n_1) = ({}^{@}f)(x)$ by [8, (49)]. □

(15) If $x \in \operatorname{Support}{}^{@}f$, then $({}^{@}f)(x) = ({}^{@}\sum \text{F-idxseq}(f))(x)$.

PROOF: Consider o being an object such that $o \in \operatorname{dom}(\mathrm{CFS}(\operatorname{Support}{}^{@}f))$ and $x = (\mathrm{CFS}(\operatorname{Support}{}^{@}f))(o)$. Reconsider $n_1 = o$ as a natural number. For every natural number i such that $i \in \operatorname{dom}(\mathrm{ev}(\text{F-idxseq}(f),x))$ and $i \neq n_1$ holds $(\mathrm{ev}(\text{F-idxseq}(f),x))(i) = 0_R$ by [8, (3), (49)]. Consider k being a natural number such that $x = (\mathrm{CFS}(\operatorname{Support}{}^{@}f))(k)$ and $k \in \operatorname{dom}(\mathrm{CFS}(\operatorname{Support}{}^{@}f))$ and $(\mathrm{ev}(\text{F-idxseq}(f),x))(k) = ({}^{@}f)(x)$. $\sum \mathrm{ev}(\text{F-idxseq}(f),x) = (\mathrm{ev}(x,R))(\sum \text{F-idxseq}(f))$. □

Now we state the propositions:

(16) Main Result: Every element of Free_LMod(X, R) is the sum of its canonical finite decomposition:
Let us consider an element f of Free-LMod(X, R). Then $f = \sum$ F-idxseq(f). The theorem is a consequence of (15) and (13).

(17) Let us consider elements f, g of Free-LMod(X, R). Suppose Support ${}^{@}f$ misses Support ${}^{@}g$. Then Support ${}^{@}(f + g) =$ Support ${}^{@}f \cup$ Support ${}^{@}g$. The theorem is a consequence of (9).

Let us consider X, R, and N. Let u be a function from X into N. The functor ${}^{@}u$ yielding an element of $(\Omega_N)^X$ is defined by the term

(Def. 25) u.

The functor P2R(u) yielding an element of Func-Mod(X, N) is defined by the term

(Def. 26) u.

Let u be an element of Func-Mod(X, N). The functor P2R(u) yielding a function from X into N is defined by the term

(Def. 27) u.

3. Universal Mapping Property

From now on B, B_1, B_2 denote elements of Fin X, f denotes an element of Free-LMod(X, R), and v denotes a function from X into the carrier of N.

Let R be a non degenerated ring, X be a non empty set, N be a left module over R, v be a function from X into the carrier of N, and f be an element of Free-LMod(X, R). The functor TermFS(f, v) yielding a function from X into the carrier of N is defined by

(Def. 28) for every element x of X, $it(x) = ({}^{@}f)(x) \cdot v(x)$.

Let B be an element of Fin X. The functor PhiSet(v, f, B) yielding an element of N is defined by the term

(Def. 29) (the addition of N)-$\sum_B$ TermFS(f, v).

Now we state the proposition:

(18) Let us consider an element x of X. Then PhiSet$(v, f, \{x\}_f) = ({}^{@}f)(x) \cdot v(x)$.

Let R be a non degenerated ring, X be a non empty set, and f be an element of Free-LMod(X, R). The functor SuppFin(f) yielding an element of Fin X is defined by the term

(Def. 30) Support ${}^{@}f$.

We define the induced homomorphism $\Phi(v) : \mathrm{FS\_Funcs}(X, R) \to N$.

Let R be a non degenerated ring, X be a non empty set, N be a left module over R, and v be a function from X into the carrier of N. The functor Φ_v yielding a function from Free-LMod(X, R) into the carrier of N is defined by

(Def. 31) for every element f of Free-LMod(X, R), $it(f) = \mathrm{PhiSet}(v, f, \mathrm{SuppFin}(f))$.

Now we state the proposition:

(19) Let us consider a non degenerated ring R, a non empty set X, a left module N over R, a function v from X into the carrier of N, and an element x of X. Then $(\Phi_v)(\mathrm{Iota}(x, R)) = v(x)$. The theorem is a consequence of (18).

Let us consider a non degenerated ring R, a non empty set X, a left module N over R, a function v from X into the carrier of N, and elements f, g of Free-LMod(X, R). Now we state the propositions:

(20) If Support $^{@}f$ misses Support $^{@}g$, then $(\Phi_v)(f + g) = (\Phi_v)(f) + (\Phi_v)(g)$. The theorem is a consequence of (9) and (17).

(21) $(\Phi_v)(f + g) = (\Phi_v)(f) + (\Phi_v)(g)$.

Now we state the propositions:

(22) Let us consider a non degenerated ring R, a non empty set X, a left module N over R, a function v from X into the carrier of N, an element f of Free-LMod(X, R), and an element a of R. Then $(\Phi_v)(a \cdot f) = a \cdot (\Phi_v)(f)$.

(23) Let us consider a non empty additive loop structure M, a finite sequence p of elements of the carrier of M, and a natural number k. If $k \in \mathrm{dom}\, p$, then $\mathrm{Del}(p, k, k) = p_{\restriction k}$.

Proof: Reconsider $p_0 = p$ as a finite sequence of elements of the carrier of M. For every natural number i such that $i \in \mathrm{dom}(\mathrm{Del}(p, k, k))$ holds $(\mathrm{Del}(p, k, k))(i) = (p_{0\restriction k})(i)$ by [16, (109)], [3, (1)], [16, (25), (112), (110)]. □

Let R be a ring, X be a set, M be a left module over R, and u be a function from X into the carrier of M. We say that u is universal if and only if

(Def. 32) for every left module N over R and for every function v from X into the carrier of N, there exists a homomorphism F from M to N by id_R such that $F \cdot u = v$ and for every homomorphisms f_1, f_2 from M to N by id_R such that $f_1 \cdot u = v$ and $f_2 \cdot u = v$ holds $f_1 = f_2$.

Let us consider a non degenerated ring R, a non empty set X, a left module N over R, and a function v from X into the carrier of N. Now we state the propositions:

(24) Φ_v is a homomorphism from Free-LMod(X, R) to N by id_R. The theorem is a consequence of (21) and (22).

(25) $(\Phi_v) \cdot (\mathrm{IotaMap}(X, R)) = v$.
PROOF: For every element x of X, $((\Phi_v) \cdot (\mathrm{IotaMap}(X, R)))(x) = v(x)$ by [9, (15)], (19). □

Let R be a ring, M, N be left modules over R, h be a function from the carrier of M into the carrier of N, and s be a finite sequence of elements of the carrier of M. One can check that $h \cdot s$ is finite sequence-like and (the carrier of N)-valued.

Now we state the propositions:

(26) Let us consider non empty sets D, E, a function h from D into E, a finite sequence p of elements of D, and an element d of D. Then $h \cdot (p \frown \langle d \rangle) = h \cdot p \frown \langle h(d) \rangle$.
PROOF: Set $q = h \cdot (p \frown \langle d \rangle)$. Set $r = h \cdot p \frown \langle h(d) \rangle$. Reconsider $p_1 = p \frown \langle d \rangle$ as a finite sequence of elements of D. For every object j such that $j \in \operatorname{dom} q$ holds $(h \cdot p_1)(j) = r(j)$ by [7, (7)], [8, (13), (12)]. □

(27) Let us consider non empty, add-associative, right zeroed, right complementable additive loop structures M, N, a function h from the carrier of M into the carrier of N, and a finite sequence s of elements of the carrier of M. If h is additive, then $h(\sum s) = \sum h \cdot s$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence s of elements of the carrier of M such that $\operatorname{len} s = \$_1$ holds $h(\sum s) = \sum h \cdot s$. $\mathcal{P}[0]$ by [18, (75), (43)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [3, (59)], [2, (11)], [17, (15)], [7, (11)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

Now we state the proposition:

(28) MAIN RESULT: THE MAP $\mathrm{IotaMap}(X, R)$ SATISFIES THE UNIVERSAL MAPPING PROPERTY OF THE FREE MODULE ON X:
Let us consider a non degenerated ring R, and a non empty set X. Then $\mathrm{IotaMap}(X, R)$ is universal. The theorem is a consequence of (24) and (25).

4. LINEAR INDEPENDENCE AND BASIS

In the sequel a denotes a natural number and p denotes a finite sequence.

Now we state the proposition:

(29) Let us consider a non degenerated ring R, a non empty set X, and an element f of $\mathrm{Free\text{-}LMod}(X, R)$. Then $f \in \mathrm{Lin}(\operatorname{rng} \mathrm{IotaMap}(X, R))$.
PROOF: Set $s = \text{F-idxseq}(f)$. Set $W = \mathrm{Lin}(\operatorname{rng} \mathrm{IotaMap}(X, R))$. $\sum s \in W$ by [18, (75)], [19, (17)], [3, (15)], [2, (11)]. □

Let us consider a non degenerated ring R and a non empty set X. Now we state the propositions:

(30) $\mathrm{Lin}(\mathrm{rng}\,\mathrm{IotaMap}(X, R)) =$ the vector space structure of Free-LMod(X, R). The theorem is a consequence of (29).

(31) $\mathrm{rng}\,\mathrm{IotaMap}(X, R)$ is linearly independent.
PROOF: Set $V =$ Free-LMod(X, R). Set $A = \mathrm{rng}\,\mathrm{IotaMap}(X, R)$. For every linear combination l of A such that $\sum l = 0_V$ holds the support of $l = \emptyset$ by [7, (11)], [16, (29)], (9), (10). □

(32) $\mathrm{rng}\,\mathrm{IotaMap}(X, R)$ is base.

Now we state the proposition:

(33) AS A COROLLARY OF (34), USING THE BASIS-THEORETIC DEFINITION OF FREE MODULES:
Let us consider a non degenerated ring R, and a non empty set X. Then Free-LMod(X, R) is free. The theorem is a consequence of (32).

5. The Free Module Generated by the Empty Set

Now we state the propositions:

(34) FS-Funcs$(\emptyset, R) = {}_R\Theta$.

(35) Free-LMod$(\emptyset, R)$ is free. The theorem is a consequence of (34).

(36) Let us consider a non degenerated ring R, and a set X. Then Free-LMod(X, R) is free. The theorem is a consequence of (33) and (35).

Let R be a non degenerated ring and X be a set. Let us note that Free-LMod(X, R) is free.

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