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General Linear Group over Arbitrary Rings

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Summary. We formalize the general linear group over an arbitrary (unital, associative) ring R as the group of units for the matrix ring over R. We loosely follow Bourbaki [6].

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Introduction

Matrix groups are just subgroups of the group of units for a ring of matrices. We formalize the notion of a `strict` matrix group over an arbitrary associative unital ring. This article is the first in the (informal, unfunded) "Lie Miz" project to formalize Lie theory in Mizar.

The first four sections contain preliminary results: a few simple facts about ring morphisms as well as centers of groups and rings, isomorphic groups having isomorphic centers, and commutative rings being equal to their centers.

We also note that we redefine the type for the center $Z(R)$ of a ring R to be a subring of R. Initially we had hoped to modify the definition in the MML, but the ordering of articles placed the definition of $Z(R)$ *before* the definition of subrings of R.

Section 5 formalizes the canonical basis for $m \times n$ matrices, which Bourbaki denoted by $\mathbf{E}_{i,j}$ and defined to be zero in all its entries except in row i and column j where it is one. The name and notation reflects the fact that the module of $m \times n$ matrices over a ring R has the $\mathbf{E}_{i,j}$ form its canonical basis.

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We call such $\mathbf{E}_{i,j}$ matrices `canMat(`i`, `j`, `m`, `n`, `R`)` where R is the scalar ring underlying the $m \times n$ matrices in question.

The family of matrices $I + \lambda \cdot \mathbf{E}_{i,j}$, where $\lambda \in R$ and $i \neq j$, gives us a rich collection of examples of invertible matrices. We provide a stockpile of results concerning them, since they turn out crucial for determining the center of matrix rings and matrix groups.

Section 6 presents the attribute (n, L)`-matrix_membered` for sets and structures whose elements are precisely $n \times n$ matrices with entries in L.

Section 7 introduces rings of square matrices $M_n(R)$, which are just the collection $n \times n$ matrices over a ring R equipped with the double loop structure using matrix addition and multiplication.

When R is a ring and S is a subring of R, it's obvious to humans that the matrix ring $M_n(S)$ is a subring of $M_n(R)$. Section 8 proves this "extension of scalars" result.

Section 9 concerns the center $Z(M_n(R))$ of a matrix ring $M_n(R)$. Recall $Z(M_n(R))$ consists of $n \times n$ scalar matrices $z \cdot I$ provided $z \in Z(R)$.

Then, in section 10, we formalize the notion of the group of units for a multiplicative loop. The group of units for a skew-field coincides with the skew-field's multiplicative group as formalized in Arneson and Rudnicki [2].

Section 11 focuses on the the group of units for the matrix ring $M_n(R)$, which is precisely the general linear group $\mathrm{GL}_n(R)$ consisting of all invertible $n \times n$ matrices with entries in the ring R. This is not yet a topological group. It remains future work to formalize $\mathrm{GL}_n^{\mathrm{top}}(R)$ when R is a topological ring.

Curiously, there is little attention in the literature given to $\mathrm{GL}_n(R)$ when R is not a field (much less when R is a noncommutative ring). Although we could spend considerable time in this setting alone, we want to provide a reasonable bedrock of results for others to use in their research.

In Section 12, we formalize the center $Z(\mathrm{GL}_n(R))$. This was rather quirky and deserves its own discussion.

Section 13 discusses the extension of scalars for the general linear group, analogous to what we did in section 8 for matrix rings.

We have three appendices concerning tangential matters.

In Appendix 1, since we were formalizing matrix addition and multiplication as binary operators of $R^{n\times n}$, we went farther and formalized matrix inversion as a unary operator of a subset of $R^{n\times n}$.

Anticipating the formalization of the topological and Lie groups (both confusingly denoted $\mathrm{GL}_n(R)$ in the literature), the remaining two Appendices concern attributes "(n, R)`-linear`" for matrix-membered groups, and "`general`" for (n, R)`-linear` groups. Thus we enable future work concerning "the general (n, R)-linear topological group" and "(n, F)-linear Lie groups".

There are a few curious aspects to this formalization worth discussing.

Weakening notions to semirings. The same constructions in this article generalize to a semiring or "rig" (i.e., a ring without subtraction — an `Abelian add-associative right_zeroed associative well-unital distributive non empty doubleLoopStr`) if the user generalizes results from the `MATRIX` series of articles. It would be hard to generalize results to anything weaker (a "double magma"?), since we cannot have an identity matrix over a nonunital "double magma".

Why would anyone want to consider a matrix ring over a semiring? We could consider the semiring of vector spaces over a field k, with the direct sum as its "addition" operator and the tensor product as its "multiplication", the zero-dimensional vector space over k is its "zero" and the one-dimensional vector space is its "one". Such things appear in representation theory, for example.

The center of the general linear group. Of particular note, seldom discussed in the literature, is that the center of $GL_1(R)$ is qualitatively different in character than $GL_n(R)$ for $n \neq 1$, specifically when we have a noncommutative ring R. This is because the general linear group of degree one is isomorphic to the group of units $GL_1(R) \cong R^\times$, so their centers must be isomorphic $Z(GL_1(R)) \cong Z(R^\times)$. But for $n \neq 1$, we have $Z(GL_n(R)) \cong (Z(R) \cap R^\times)$. When R is commutative, $Z(R) = R$ and $Z(R^\times) = R^\times$, so there is no discrepency.

For noncommutative R, this is not always the case. Consider a ring R and a ring *endo*-morphism $\sigma\colon R \to R$, then construct the "Twisted" polynomial ring $R[x;\sigma]$ where each $a \in R$ obeys $xa = \sigma(a) \cdot x$. This twisted polynomial ring has $Z(R^\times) \setminus (Z(R) \cap R^\times) \neq \emptyset$, as discussed in Example 1.7 of Lam [15]. When $R = \mathbb{C}$ and σ is complex conjugation, we find $Z(R) = \mathbb{R}[x]$ and $R^\times = \mathbb{C}^\times$ gives us $Z(R) \cap R^\times \cong \mathbb{R}^\times$, but $Z(R) = \mathbb{C}^\times$. Such an example may be worth formalizing in the future.

To the best of the author's knowledge, this is the first article discussing such results. The overwhelming majority of the literature generalizes only so far as to the setting where R is a commutative ring.

1. Results about Ring Morphisms

From now on R denotes a ring, L denotes an associative, right unital, non empty multiplicative loop structure, n denotes a natural number, and S denotes a subring of R.

Now we state the propositions:

(1) Let us consider rings R_1, R_2, and a function φ from R_1 into R_2. If φ is linear, then $\varphi(1_{R_1}) = 1_{R_2}$.

(2) Let us consider rings R_1, R_2. Suppose R_1 and R_2 are isomorphic. Then

(i) R_2 is R_1-isomorphic, and

(ii) R_1 is R_2-isomorphic.

(3) Let us consider rings R_1, R_2, and a function φ from R_1 into R_2. Suppose φ is isomorphism. Let us consider a function ψ from R_2 into R_1. If $\psi = \varphi^{-1}$, then ψ is bijective.

(4) Let us consider rings R_1, R_2, and a function φ from R_1 into R_2. Suppose φ is isomorphism. Let us consider an element x of R_1. If $\varphi(x) = 1_{R_2}$, then $x = 1_{R_1}$. The theorem is a consequence of (1).

2. Centers of Groups and Rings

Now we state the proposition:

(5) Let us consider groups G_1, G_2. If G_1 and G_2 are isomorphic, then $\mathrm{Z}(G_1)$ and $\mathrm{Z}(G_2)$ are isomorphic.
PROOF: Consider φ being a homomorphism from G_1 to G_2 such that φ is bijective. There exists a homomorphism ψ from $\mathrm{Z}(G_1)$ to $\mathrm{Z}(G_2)$ such that ψ is bijective by [10, (52)], [11, (32)], [10, (49)], [19, (77)]. □

Let us consider R. Let us note that there exists a subring of R which is strict, commutative, and non empty.

Let us note that the functor $\mathrm{Z}(R)$ yields a strict, commutative, non empty subring of R. Now we state the propositions:

(6) Let us consider a commutative ring R, and an object x. If $x \in R$, then $x \in \mathrm{Z}(R)$.

(7) Let us consider a commutative ring R, and an element x of R. Then $x \in \mathrm{Z}(R)$.

(8) Let us consider a commutative ring R. Then the carrier of $\mathrm{Z}(R)$ = the carrier of R. The theorem is a consequence of (7).

(9) Let us consider a strict, commutative ring R. Then $\mathrm{Z}(R) = R$. The theorem is a consequence of (8).

3. Some Results about Matrix Operations

Let us consider n and R. Let A, B be square matrices over R of dimension n. Let us observe that the functor $A \cdot B$ yields a square matrix over R of dimension n. Now we state the propositions:

(10) Let us consider a natural number m, and a matrix A over R. Suppose $\mathrm{width}\, 0_R^{m \times (\mathrm{len}\, A)} = \mathrm{len}\, A$. Then $0_R^{m \times (\mathrm{len}\, A)} \cdot A = 0_R^{m \times (\mathrm{width}\, A)}$.

(11) Let us consider a natural number n, a ring R, and a matrix A over R. Suppose $n = \operatorname{width} 0_R^{(\operatorname{width} A)\times n}$. Then $A \cdot (0_R^{(\operatorname{width} A)\times n}) = 0_R^{(\operatorname{len} A)\times n}$.

Let us consider natural numbers i, k, m, n and a matrix A over R of dimension $m\times n$. Now we state the propositions:

(12) If $i \in \operatorname{dom}(0_R^{m\times m})$, then $\operatorname{Line}(0_R^{m\times m}, i) \cdot A_{\square,k} = 0_R$.
Proof: Set $p = \operatorname{Line}(0_R^{m\times m}, i) \bullet A_{\square,k}$. For every element i' of $\mathbb{N}$ such that $i' \in \operatorname{dom} p$ holds $p(i') = 0_R$ by [24, (13)], [12, (87)], [13, (1)], [23, (60)]. □

(13) If $k \in \operatorname{dom}(0_R^{n\times n})$, then $\operatorname{Line}(A, i) \cdot (0_R^{n\times n})_{\square,k} = 0_R$.
Proof: Set $p = \operatorname{Line}(A, i) \bullet (0_R^{n\times n})_{\square,k}$. For every element i' of $\mathbb{N}$ such that $i' \in \operatorname{dom} p$ holds $p(i') = 0_R$ by [24, (13)], [12, (87)], [13, (1)], [23, (60)]. □

(14) If $i \in \operatorname{dom}(I_R^{m\times m})$ and $i \in \operatorname{dom} A$, then $\operatorname{Line}(I_R^{m\times m}, i) \cdot A_{\square,k} = A_{i,k}$.
Proof: Set $p = \operatorname{Line}(I_R^{m\times m}, i)$. Set $q = A_{\square,k}$. $i \in \operatorname{dom} p$ and $i \in \operatorname{dom} q$ and $p(i) = 1_R$ and for every natural number k such that $k \in \operatorname{dom} p$ and $k \neq i$ holds $p(k) = 0_R$ by [12, (87)]. □

(15) If $k \in \operatorname{dom} A$ and $k \in \operatorname{dom}(I_R^{n\times n})$, then $\operatorname{Line}(A, i) \cdot (I_R^{n\times n})_{\square,k} = A_{i,k}$.

Now we state the proposition:

(16) Let us consider natural numbers k, m. Suppose if $k = 0$, then $m = 0$. Let us consider an element r of R, a matrix A over R of dimension $k\times m$, and a matrix B over R of dimension $m\times n$. Then $A \cdot (r \cdot B) = (A \cdot r) \cdot B$.

Let us consider elements r_1, r_2 of R and a matrix A over R. Now we state the propositions:

(17) $(r_1 \cdot A) \cdot r_2 = r_1 \cdot (A \cdot r_2)$.

(18) $r_1 \cdot (r_2 \cdot A) = (r_1 \cdot r_2) \cdot A$.

Now we state the propositions:

(19) Let us consider an element r of R. Then $r \cdot (0_R^{n\times n}) = 0_R^{n\times n}$.
Proof: For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of $0_R^{n\times n}$ holds $r \cdot (0_R^{n\times n})_{i,j} = 0_R^{n\times n}{}_{i,j}$ by [13, (1)]. □

(20) Let us consider natural numbers k, m. Suppose if $k = 0$, then $m = 0$. Let us consider an element r of R, a matrix A over R of dimension $k\times m$, and a matrix B over R of dimension $m\times n$. Then $(r \cdot A) \cdot B = r \cdot (A \cdot B)$.

(21) Let us consider a matrix A over R, and an element r of R. Then $(-r)\cdot A = -r \cdot A$.
Proof: $\operatorname{len}(-r) \cdot A = \operatorname{len} A$ and $\operatorname{len}(-r \cdot A) = \operatorname{len} A$ and $\operatorname{len} r \cdot A = \operatorname{len} A$. $\operatorname{width}(-r) \cdot A = \operatorname{width} A$ and $\operatorname{width}(-r \cdot A) = \operatorname{width} A$ and $\operatorname{width} r \cdot A = \operatorname{width} A$. For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of $(-r) \cdot A$ holds $(-r) \cdot A_{i,j} = (-r \cdot A)_{i,j}$ by [14, (9)]. □

(22) Let us consider a matrix A over R. Then $1_R \cdot A = A$.

4. Results about 1-by-1 Matrices

Now we state the propositions:

(23) Let us consider a non empty set D, and a square matrix M over D of dimension 1. Then $M = \langle\langle M_{1,1}\rangle\rangle$.
PROOF: For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = \langle\langle M_{1,1}\rangle\rangle_{i,j}$ by [12, (87)], [5, (2)]. □

(24) Let us consider a non empty zero-one structure L. Then $I_L^{1\times 1} = \langle\langle 1_L\rangle\rangle$. The theorem is a consequence of (23).

Let us consider elements x, y of R and square matrices M_1, M_2 over R of dimension 1. Now we state the propositions:

(25) If $M_1 = \langle\langle x\rangle\rangle$ and $M_2 = \langle\langle y\rangle\rangle$, then $M_1 \cdot M_2 = \langle\langle x \cdot y\rangle\rangle$.

(26) If $M_1 = \langle\langle x\rangle\rangle$ and $M_2 = \langle\langle y\rangle\rangle$, then $M_1 + M_2 = \langle\langle x + y\rangle\rangle$.
PROOF: For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of $M_1 + M_2$ holds $(M_1 + M_2)_{i,j} = \langle\langle x + y\rangle\rangle_{i,j}$ by [12, (87)], [5, (2)]. □

Now we state the propositions:

(27) Let us consider elements r, s of R. Then

(i) $r \cdot \langle\langle s\rangle\rangle = \langle\langle r \cdot s\rangle\rangle$, and

(ii) $\langle\langle s\rangle\rangle \cdot r = \langle\langle s \cdot r\rangle\rangle$.

PROOF: Set $M = \langle\langle s\rangle\rangle$. Set $M_1 = \langle\langle r \cdot s\rangle\rangle$. Set $M_2 = \langle\langle s \cdot r\rangle\rangle$. For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of $r \cdot M$ holds $r \cdot M_{i,j} = M_{1i,j}$ by [5, (2)], [12, (87)]. For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of $M \cdot r$ holds $M \cdot r_{i,j} = M_{2i,j}$ by [12, (87)], [5, (2)]. □

(28) Let us consider a square matrix Y over R of dimension 1. Then $Y = Y_{1,1} \cdot (I_R^{1\times 1})$.
PROOF: For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of Y holds $Y_{i,j} = Y_{1,1} \cdot (I_R^{1\times 1})_{i,j}$ by [12, (87)], [5, (2)]. □

5. Canonical Basis for Matrices

Let R be a non empty zero-one structure and I, K, i, k be natural numbers. The functor $\mathrm{canMat}(i, k, I, K, R)$ yielding a matrix over R of dimension $I \times K$ is defined by

(Def. 1) for every natural numbers i_2, k_2 such that $\langle i_2, k_2\rangle \in$ the indices of *it* holds if $\langle i, k\rangle = \langle i_2, k_2\rangle$, then $it_{i_2,k_2} = 1_R$ and if $\langle i, k\rangle \neq \langle i_2, k_2\rangle$, then $it_{i_2,k_2} = 0_R$.

Now we state the propositions:

(29) Let us consider natural numbers I, K, i, k, i_2, k_2. Suppose $\langle i_2, k_2\rangle \in$ the indices of $\mathrm{canMat}(i,k,I,K,R)$. Then

(i) if $i \neq i_2$, then $(\mathrm{canMat}(i,k,I,K,R))_{i_2,k_2} = 0_R$, and

(ii) if $k \neq k_2$, then $(\mathrm{canMat}(i,k,I,K,R))_{i_2,k_2} = 0_R$, and

(iii) if $i = i_2$ and $k = k_2$, then $(\mathrm{canMat}(i,k,I,K,R))_{i_2,k_2} = 1_R$.

(30) Let us consider natural numbers I, K, i, k. Suppose $\langle i, k\rangle \notin$ the indices of $\mathrm{canMat}(i,k,I,K,R)$. Then $\mathrm{canMat}(i,k,I,K,R) = 0_R^{I\times K}$.
Proof: For every natural numbers i_2, k_2 such that $\langle i_2, k_2\rangle \in$ the indices of $\mathrm{canMat}(i,k,I,K,R)$ holds $(\mathrm{canMat}(i,k,I,K,R))_{i_2,k_2} = {0_R^{I\times K}}_{i_2,k_2}$ by (29), [24, (1)]. $\square$

(31) Let us consider a non empty zero-one structure R, and natural numbers I, K. Suppose $I = 0$ iff $K = 0$. Let us consider natural numbers i, k. Then $(\mathrm{canMat}(i,k,I,K,R))^{\mathrm{T}} = \mathrm{canMat}(k,i,K,I,R)$.

(32) Let us consider natural numbers i, k, I, K. Then

(i) $\mathrm{len}\,\mathrm{canMat}(i,k,I,K,R) = I$, and

(ii) if $I = 0$, then $\mathrm{width}\,\mathrm{canMat}(i,k,I,K,R) = 0$, and

(iii) if $I \neq 0$, then $\mathrm{width}\,\mathrm{canMat}(i,k,I,K,R) = K$.

(33) Let us consider natural numbers I, K, i, k, i_2. Suppose $\langle i_2, k\rangle \in$ the indices of $\mathrm{canMat}(i,k,I,K,R)$. Then

(i) if $i_2 = i$, then $\mathrm{Line}(\mathrm{canMat}(i,k,I,K,R), i_2) = \mathrm{Line}(I_R^{K\times K}, k)$, and

(ii) if $i_2 \neq i$, then $\mathrm{Line}(\mathrm{canMat}(i,k,I,K,R), i_2) = \mathrm{Line}(0_R^{K\times K}, k)$.

Proof: Set $p = \mathrm{Line}(\mathrm{canMat}(i,k,I,K,R), i_2)$. For every natural number k' such that $k' \in \mathrm{dom}\, p$ holds $\langle i_2, k'\rangle \in$ the indices of $\mathrm{canMat}(i,k,I,K,R)$ by [12, (87)]. $\square$

(34) Let us consider natural numbers I, K, i, k, k_2. Suppose $\langle i, k_2\rangle \in$ the indices of $\mathrm{canMat}(i,k,I,K,R)$. Then

(i) if $k_2 = k$, then $(\mathrm{canMat}(i,k,I,K,R))_{\square,k_2} = (I_R^{I\times I})_{\square,i}$, and

(ii) if $k_2 \neq k$, then $(\mathrm{canMat}(i,k,I,K,R))_{\square,k_2} = (0_R^{I\times I})_{\square,i}$.

Proof: $i \in \mathrm{dom}(\mathrm{canMat}(i,k,I,K,R))$ and $k_2 \in \mathrm{Seg}\,\mathrm{width}\,\mathrm{canMat}(i,k,I,K,R)$ by [12, (87)]. For every natural number j' such that $j' \in \mathrm{dom}((\mathrm{canMat}(i,k,I,K,R))_{\square}$ holds $\langle j', k_2\rangle \in$ the indices of $\mathrm{canMat}(i,k,I,K,R)$ by [12, (87)]. For every natural number j' such that $j' \in \mathrm{dom}((\mathrm{canMat}(i,k,I,K,R))_{\square,k_2})$ holds $\langle j', i\rangle \in$ the indices of $I_R^{I\times I}$ by [12, (87)]. If $k_2 = k$, then $(\mathrm{canMat}(i,k,I,K,R))_{\square,k_2} =$ $(I_R^{I\times I})_{\square,i}$ by [12, (87)], (29), [5, (13)]. For every natural number j' such that $j' \in \mathrm{dom}((\mathrm{canMat}(i,k,I,K,R))_{\square,k_2})$ holds $\langle j', i\rangle \in$ the indices of $0_R^{I\times I}$ by [12, (87)]. For every natural number j' such that $j' \in \mathrm{dom}((\mathrm{canMat}(i,k,I,K,R))_{\square,k_2})$

holds $((\mathrm{canMat}(i,k,I,K,R))_{\square,k_2})(j') = ((0_R^{I\times I})_{\square,i})(j')$ by [12, (87)], (29), [13, (1)]. $\square$

Now we state the proposition:

(35) Bourbaki, Algebra [6], II 10.3 equation (9):
Let us consider natural numbers I, K, L. Suppose $I \neq 0$ and $K \neq 0$. Let us consider natural numbers i, k_1, k_2, l. Suppose $k_1 \neq k_2$. Then $(\mathrm{canMat}(i,k_1,I,K,R)) \cdot (\mathrm{canMat}(k_2,l,K,L,R)) = 0_R^{I\times L}$. The theorem is a consequence of (30), (10), (11), (33), (14), (29), and (12).

Now we state the propositions:

(36) Bourbaki, Algebra [6], II 10.3 equation (9):
Let us consider natural numbers I, K, L, i, k, l. Suppose $i \in \mathrm{Seg}\, I$ and $k \in \mathrm{Seg}\, K$ and $l \in \mathrm{Seg}\, L$. Then $(\mathrm{canMat}(i,k,I,K,R))\cdot(\mathrm{canMat}(k,l,K,L,R)) = \mathrm{canMat}(i,l,I,L,R)$. The theorem is a consequence of (32), (33), (12), (29), (34), (13), and (14).

(37) Let us consider natural numbers I, K, L. Suppose if $I = 0$, then $K = 0$. Let us consider natural numbers i, k, l. Suppose $k \in \mathrm{Seg}\, K$. Then $(\mathrm{canMat}(i,k,I,K,R))\cdot(\mathrm{canMat}(k,l,K,L,R)) = \mathrm{canMat}(i,l,I,L,R)$. The theorem is a consequence of (36), (30), (10), (32), and (11).

Let us consider R and n. Let i, k be natural numbers. The functor $\mathrm{canMat}(i,k,n,R)$ yielding a square matrix over R of dimension n is defined by the term

(Def. 2) $\mathrm{canMat}(i,k,n,n,R)$.

Now we state the propositions:

(38) Let us consider natural numbers i, k, i_2, k_2. Suppose $\langle i_2, k_2\rangle \in$ the indices of $\mathrm{canMat}(i,k,n,R)$. Then

(i) if $i \neq i_2$, then $(\mathrm{canMat}(i,k,n,R))_{i_2,k_2} = 0_R$, and

(ii) if $k \neq k_2$, then $(\mathrm{canMat}(i,k,n,R))_{i_2,k_2} = 0_R$, and

(iii) if $i = i_2$ and $k = k_2$, then $(\mathrm{canMat}(i,k,n,R))_{i_2,k_2} = 1_R$.

(39) Let us consider natural numbers I, i, j_1, j_2, k. Suppose $j_1 \neq j_2$. Then $(\mathrm{canMat}(i,j_1,I,R)) \cdot (\mathrm{canMat}(j_2,k,I,R)) = 0_R^{I\times I}$. The theorem is a consequence of (35).

(40) Let us consider natural numbers i, j. Suppose $\langle i, j\rangle \notin$ the indices of $\mathrm{canMat}(i,j,n,R)$. Then $\mathrm{canMat}(i,j,n,R) = 0_R^{n\times n}$.
Proof: For every natural numbers i', j' such that $\langle i', j'\rangle \in$ the indices of $0_R^{n\times n}$ holds ${0_R^{n\times n}}_{i',j'} = {0_R^{n\times n}}_{i',j'}$ by [13, (1)], [24, (1)]. $\square$

(41) Let us consider natural numbers i, k, l. Suppose $k \in \mathrm{Seg}\, n$. Then $(\mathrm{canMat}(i,k,n,R)$ $(\mathrm{canMat}(k,l,n,R)) = \mathrm{canMat}(i,l,n,R)$.

Let us consider natural numbers i, j, a square matrix A over R of dimension n, and natural numbers i_2, j_2. Now we state the propositions:

(42) Suppose $\langle i, j\rangle \in$ the indices of $\mathrm{canMat}(i, j, n, R)$. Then suppose $\langle i_2, j_2\rangle \in$ the indices of A. Then

(i) if $i = i_2$, then $(\mathrm{canMat}(i, j, n, R)) \cdot A_{i_2,j_2} = A_{j,j_2}$, and

(ii) if $i \neq i_2$, then $(\mathrm{canMat}(i, j, n, R)) \cdot A_{i_2,j_2} = 0_R$.

Proof: $\langle i_2, j\rangle \in$ the indices of $\mathrm{canMat}(i, j, n, R)$ by [12, (87)]. If $i = i_2$, then $(\mathrm{canMat}(i, j, n, R)) \cdot A_{i_2,j_2} = A_{j,j_2}$ by [12, (87)], (33), (14). $j \in \mathrm{dom}(0_R^{n\times n})$ by [12, (87)]. □

(43) Suppose $\langle i, j\rangle \in$ the indices of $\mathrm{canMat}(i, j, n, R)$. Then suppose $\langle i_2, j_2\rangle \in$ the indices of A. Then

(i) if $j = j_2$, then $A \cdot (\mathrm{canMat}(i, j, n, R))_{i_2,j_2} = A_{i_2,i}$, and

(ii) if $j \neq j_2$, then $A \cdot (\mathrm{canMat}(i, j, n, R))_{i_2,j_2} = 0_R$.

Proof: $\langle i, j_2\rangle \in$ the indices of $\mathrm{canMat}(i, j, n, R)$ by [12, (87)]. $i \in \mathrm{dom}\, A$ and $i \in \mathrm{dom}(I_R^{n\times n})$ by [12, (87)]. $\langle i_2, j_2\rangle \in$ the indices of $A\cdot(\mathrm{canMat}(i, j, n, R))$. If $j = j_2$, then $A \cdot (\mathrm{canMat}(i, j, n, R))_{i_2,j_2} = A_{i_2,i}$. □

Now we state the propositions:

(44) Let us consider an element r of R, and natural numbers i, j. Then $(\mathrm{canMat}(i, j, n, R)) \cdot r = r \cdot (\mathrm{canMat}(i, j, n, R))$.

(45) Let us consider natural numbers i, j. Suppose $i \neq j$. Let us consider an element r of R. Then $I_R^{n\times n} + r \cdot (\mathrm{canMat}(i, j, n, R))$ is an invertible square matrix over R of dimension n. The theorem is a consequence of (16), (17), (44), (18), (20), (39), and (19).

(46) Let us consider natural numbers i, j, an element r of R, and a square matrix A over R of dimension n. Then $A\cdot(I_R^{n\times n}+r\cdot(\mathrm{canMat}(i, j, n, R))) = (I_R^{n\times n}+r\cdot(\mathrm{canMat}(i, j, n, R)))\cdot A$ if and only if $A\cdot(r\cdot(\mathrm{canMat}(i, j, n, R))) = (r \cdot (\mathrm{canMat}(i, j, n, R))) \cdot A$.

(47) Let us consider natural numbers i, j, and a square matrix A over R of dimension n. Then $A\cdot(I_R^{n\times n}+\mathrm{canMat}(i, j, n, R)) = (I_R^{n\times n}+\mathrm{canMat}(i, j, n, R))\cdot A$ if and only if $A \cdot (\mathrm{canMat}(i, j, n, R)) = (\mathrm{canMat}(i, j, n, R)) \cdot A$. The theorem is a consequence of (22) and (46).

Let us consider natural numbers i, j. Now we state the propositions:

(48) If $i > j$, then $\mathrm{canMat}(i, j, n, R)$ is lower triangular.

(49) If $i < j$, then $\mathrm{canMat}(i, j, n, R)$ is upper triangular.

Let us consider natural numbers i, j and an element r of R. Now we state the propositions:

(50) If $i > j$, then $r \cdot (\mathrm{canMat}(i, j, n, R))$ is lower triangular. The theorem is a consequence of (48).

(51) If $i < j$, then $r \cdot (\mathrm{canMat}(i, j, n, R))$ is upper triangular. The theorem is a consequence of (49).

(52) If $i > j$, then $I_R^{n \times n} + r \cdot (\mathrm{canMat}(i, j, n, R))$ is lower triangular. The theorem is a consequence of (50).

(53) If $i < j$, then $I_R^{n \times n} + r \cdot (\mathrm{canMat}(i, j, n, R))$ is upper triangular. The theorem is a consequence of (51).

(54) If $i > j$, then $(\mathrm{canMat}(i, j, n, R)) \cdot r$ is lower triangular. The theorem is a consequence of (48).

(55) If $i < j$, then $(\mathrm{canMat}(i, j, n, R)) \cdot r$ is upper triangular. The theorem is a consequence of (49).

(56) If $i > j$, then $I_R^{n \times n} + (\mathrm{canMat}(i, j, n, R)) \cdot r$ is lower triangular. The theorem is a consequence of (54).

(57) If $i < j$, then $I_R^{n \times n} + (\mathrm{canMat}(i, j, n, R)) \cdot r$ is upper triangular. The theorem is a consequence of (55).

6. Matrix-membered Structures and Sets

From now on L denotes a non empty double loop structure.

Let L be a non empty 1-sorted structure. Let us consider n. Let I_1 be a set. We say that I_1 is (n,L)-matrix-membered if and only if

(Def. 3) for every object x such that $x \in I_1$ holds x is a square matrix over L of dimension n.

Let us consider L. One can check that $L^{n \times n}$ is (n,L)-matrix-membered.

Now we state the proposition:

(58) Let us consider a non empty 1-sorted structure L. Then ((the carrier of $L)^n)^n$ is (n,L)-matrix-membered.

Let L be a non empty 1-sorted structure. Let us consider n. Observe that there exists a set which is non empty and (n,L)-matrix-membered.

Let A be a non empty, (n,L)-matrix-membered set.

Observe that an element of A is a square matrix over L of dimension n. Let I_1 be a 1-sorted structure. We say that I_1 is (n,L)-matrix-membered if and only if

(Def. 4) the carrier of I_1 is (n,L)-matrix-membered.

Now we state the proposition:

(59) Let us consider a 1-sorted structure S. Then the carrier of $S \subseteq L^{n \times n}$ if and only if S is (n,L)-matrix-membered.

Proof: For every object x such that $x \in$ the carrier of S holds $x \in L^{n \times n}$ by [13, (2)]. □

Let L be a non empty 1-sorted structure. Let us consider n. One can check that there exists a 1-sorted structure which is non empty and (n,L)-matrix-membered.

Let us consider L. One can verify that there exists a 1-sorted structure which is strict, non empty, and (n,L)-matrix-membered.

7. Matrix Rings

Let us consider n and L. The functor matrix-mult(n, L) yielding a binary operation on $L^{n \times n}$ is defined by

(Def. 5) for every elements A, B of $L^{n \times n}$, $it(A, B) = A \cdot B$.

The functor matrix-add(n, L) yielding a binary operation on $L^{n \times n}$ is defined by

(Def. 6) for every elements A, B of $L^{n \times n}$, $it(A, B) = A + B$.

We introduce the notation One(L, n) as a synonym of $I_L^{n \times n}$ and Zero(L, n) as a synonym of $0_L^{n \times n}$.

The functors: One(L, n) and Zero(L, n) yield elements of $L^{n \times n}$. The functor Matrix-ring(n, L) yielding a strict double loop structure is defined by the term

(Def. 7) $\langle L^{n \times n}, \text{matrix-add}(n, L), \text{matrix-mult}(n, L), \text{One}(L, n), \text{Zero}(L, n)\rangle$.

Note that Matrix-ring(n, L) is strict, non empty, and (n,L)-matrix-membered.

Now we state the proposition:

(60) Let us consider an object x. Then x is an element of Matrix-ring(n, L) if and only if x is a square matrix over L of dimension n.

Let us consider L. One can check that Matrix-ring$(0, L)$ is trivial.

Now we state the propositions:

(61) Let us consider square matrices A, B over L of dimension n, and elements X, Y of Matrix-ring(n, L). If $A = X$ and $B = Y$, then $A + B = X + Y$.

(62) Let us consider square matrices X, Y over L of dimension n, and elements x, y of $L^{n \times n}$. If $X = x$ and $Y = y$, then $X \cdot Y = (\text{matrix-mult}(n, L))(x, y)$.

(63) Let us consider square matrices A, B over L of dimension n, and elements X, Y of Matrix-ring(n, L). If $A = X$ and $B = Y$, then $A \cdot B = X \cdot Y$.

(64) Let us consider square matrices A, B, C over R of dimension n. Then $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

(65) $I_L^{n \times n}$ is an element of Matrix-ring(n, L).

(66) $0_L^{n \times n}$ is an element of Matrix-ring(n, L).

Let us consider n. Let R be a right complementable, Abelian, add-associative, right zeroed, non empty double loop structure. One can check that Matrix-ring(n, R) is add-associative and Matrix-ring(n, R) is right zeroed and Matrix-ring(n, R) is right complementable and Matrix-ring(n, R) is Abelian.

Let us consider R. Let us note that Matrix-ring(n, R) is well unital and Matrix-ring(n, R) is distributive and Matrix-ring(n, R) is associative.

Now we state the proposition:

(67) Matrix-ring$(1, R)$ and R are isomorphic.
PROOF: Set $U_5 =$ the carrier of R. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 = \langle\langle \$_1 \rangle\rangle$. For every element x of U_5, there exists an element y of $R^{1 \times 1}$ such that $\mathcal{P}[x, y]$ by [13, (2)]. Consider f being a function from U_5 into $R^{1 \times 1}$ such that for every element r of U_5, $\mathcal{P}[r, f(r)]$ from [11, Sch. 3]. f is onto by (60), (23), [11, (10)]. f is one-to-one by [5, (37)], [12, (3)], [21, (1)]. f is linear. □

8. Extension of Scalars for Matrix Ring

Now we state the propositions:

(68) Let us consider non empty sets D, E. If $D \subseteq E$, then every matrix over D is a matrix over E.

(69) Let us consider natural numbers m, n, and non empty sets D, E. Suppose $D \subseteq E$. Then every matrix over D of dimension $m \times n$ is a matrix over E of dimension $m \times n$. The theorem is a consequence of (68).

(70) Every square matrix over S of dimension n is a square matrix over R of dimension n. The theorem is a consequence of (69).

(71) Let us consider non empty 1-sorted structures S_1, S_2, a square matrix X over S_2 of dimension n, and a square matrix Y over S_1 of dimension n. Then for every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of X holds $X_{i,j} = Y_{i,j}$ if and only if $X = Y$.
PROOF: If for every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of X holds $X_{i,j} = Y_{i,j}$, then $X = Y$ by [12, (87)], [5, (13)]. Consider q being a finite sequence of elements of S_1 such that $q = Y(i)$ and $Y_{i,j} = q(j)$. Consider p being a finite sequence of elements of S_2 such that $p = X(i)$ and $X_{i,j} = p(j)$. □

(72) $S^{n \times n} \subseteq R^{n \times n}$.

Proof: For every object x such that $x \in S^{n \times n}$ holds $x \in R^{n \times n}$ by [13, (2)], (70). □

(73) Let us consider square matrices a, b over S of dimension n, and square matrices A, B over R of dimension n. If $a = A$ and $b = B$, then $a + b = A + B$.

Proof: Set $L_1 = a + b$. Set $R_3 = A + B$. For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of L_1 holds $L_{1i,j} = R_{3i,j}$ by (71), [18, (15)]. □

(74) matrix-add$(n, S) =$ matrix-add$(n, R) \restriction S^{n \times n}$.

Proof: Set $U_6 = S^{n \times n}$. Set $U_5 = R^{n \times n}$. $U_6 \subseteq U_5$. For every elements x, y of $S^{n \times n}$, (matrix-add$(n, S))(x, y) =$ (matrix-add$(n, R) \restriction S^{n \times n})(x, y)$ by (70), (72), (73), [12, (87)]. □

(75) Let us consider a square matrix a over S of dimension n, and a square matrix A over R of dimension n. Suppose $a = A$. Let us consider a natural number i. If $i \in \operatorname{dom} a$, then $\operatorname{Line}(a, i) = \operatorname{Line}(A, i)$.

Proof: For every natural number j such that $j \in \operatorname{dom} \operatorname{Line}(a, i)$ holds $\operatorname{Line}(a, i)(j) = \operatorname{Line}(A, i)(j)$ by [12, (87)], (71). □

(76) Let us consider a square matrix b over S of dimension n, and a square matrix B over R of dimension n. Suppose $b = B$. Let us consider a natural number j. If $j \in \operatorname{Seg} \operatorname{width} b$, then $b_{\square, j} = B_{\square, j}$.

Proof: For every natural number i such that $i \in \operatorname{dom}(b_{\square, j})$ holds $(b_{\square, j})(i) = (B_{\square, j})(i)$ by [12, (87)], (71). □

(77) Let us consider a finite sequence a of elements of S, and a finite sequence A of elements of R. If $a = A$, then $\sum a = \sum A$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence a of elements of S for every finite sequence A of elements of R such that $a = A$ and $\operatorname{len} a = \$_1$ holds $\sum a = \sum A$. $\mathcal{P}[0]$ by [20, (75)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [9, (19), (17), (16)], [20, (44), (41)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. □

(78) Let us consider finite sequences a, b of elements of S, and finite sequences A, B of elements of R. If $a = A$ and $b = B$, then $a \bullet b = A \bullet B$.

Proof: $\operatorname{dom}(a \bullet b) = \operatorname{dom}(A \bullet B)$ by [9, (71)]. For every natural numbers i, j, k such that $i \in \operatorname{Seg} \min(j, k)$ holds $i \in \operatorname{Seg} j$ and $i \in \operatorname{Seg} k$ by [9, (2)]. For every natural number i such that $i \in \operatorname{dom}(a \bullet b)$ holds $(a \bullet b)(i) = (A \bullet B)(i)$ by [9, (71)], [10, (106)], [23, (60)], [18, (16)]. □

(79) Let us consider square matrices a, b over S of dimension n, and square matrices A, B over R of dimension n. If $a = A$ and $b = B$, then $a \cdot b = A \cdot B$.

Proof: Set $L_1 = a \cdot b$. Set $R_3 = A \cdot B$. For every natural numbers i, j

such that $\langle i, j\rangle \in$ the indices of L_1 holds $L_{1i,j} = R_{3i,j}$ by [12, (87)], (75), (76), (77). □

(80) matrix-mult(n, S) = matrix-mult$(n, R) \restriction S^{n\times n}$.

PROOF: Set $U_6 = S^{n\times n}$. Set $U_5 = R^{n\times n}$. $U_6 \subseteq U_5$. For every elements x, y of $S^{n\times n}$, (matrix-mult$(n, S))(x, y)$ = (matrix-mult$(n, R) \restriction S^{n\times n})(x, y)$ by (70), (72), (79), [10, (49)]. □

(81) $0_S^{n\times n} = 0_R^{n\times n}$.

(82) $I_S^{n\times n} = I_R^{n\times n}$. The theorem is a consequence of (70) and (71).

(83) (i) One(S, n) = One(R, n), and

(ii) Zero(S, n) = Zero(R, n).

Now we state the proposition:

(84) EXTENSION OF SCALARS FOR MATRIX RINGS:

Matrix-ring(n, S) is a subring of Matrix-ring(n, R). The theorem is a consequence of (72), (74), (80), and (83).

9. CENTER OF THE MATRIX RING

Now we state the propositions:

(85) Let us consider an element z of R. If $z \in \mathrm{Z}(R)$, then for every matrix A over R, $A \cdot z = z \cdot A$.

PROOF: len $A \cdot z$ = len $z \cdot A$ and width $A \cdot z$ = width $z \cdot A$. For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of $A \cdot z$ holds $A \cdot z_{i,j} = z \cdot A_{i,j}$ by [3, (17)]. □

(86) Let us consider an element z of R. Suppose $z \in \mathrm{Z}(R)$. Let us consider square matrices A, B over R of dimension n. If $A \cdot B = B \cdot A$, then $A \cdot (z \cdot B) = (z \cdot B) \cdot A$. The theorem is a consequence of (16), (85), and (20).

(87) Let us consider an element z of R. Suppose $z \in \mathrm{Z}(R)$. Let us consider a square matrix A over R of dimension n. Then $A \cdot (z \cdot (I_R^{n\times n})) = (z \cdot (I_R^{n\times n})) \cdot A$. The theorem is a consequence of (86).

Let us consider elements x, y of R. Now we state the propositions:

(88) $x \cdot (I_R^{n\times n}) \cdot y = x \cdot y \cdot (I_R^{n\times n})$.

(89) $x \cdot (I_R^{n\times n}) \cdot (y \cdot (I_R^{n\times n})) = (x \cdot (I_R^{n\times n}) \cdot y) \cdot (I_R^{n\times n})$.

Now we state the propositions:

(90) SEE, E.G., BOURBAKI'S "ALGEBRA" VIII [7] 5 NO.3 COROLLARY 2 TO THEOREM 2:

Let us consider a square matrix A over R of dimension n. Suppose for every square matrix B over R of dimension n, $A \cdot B = B \cdot A$. Then there exists an element z of R such that

(i) $z \in \mathrm{Z}(R)$, and

(ii) $A = z \cdot (I_R^{n \times n})$.

PROOF: For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of A and $i \neq j$ holds $A_{i,j} = 0_R$ by [12, (87)], (43), (42). For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of A holds $A_{i,i} = A_{j,j}$. □

(91) Let us consider an element X of Matrix-ring(n, R), and a square matrix A over R of dimension n. Suppose $A = X$. Then $X \in \mathrm{Z}(\text{Matrix-ring}(n, R))$ if and only if for every square matrix B over R of dimension n, $A \cdot B = B \cdot A$. The theorem is a consequence of (60) and (63).

(92) Let us consider an element X of Matrix-ring(n, R). Then $X \in \mathrm{Z}(\text{Matrix-ring}(n, R))$ if and only if there exists an element z of R such that $z \in \mathrm{Z}(R)$ and $X = z \cdot (I_R^{n \times n})$. The theorem is a consequence of (60), (91), (90), (63), and (87).

10. GROUP OF UNITS FOR UNITAL RINGS

In the sequel M denotes an associative, right unital, non empty multiplicative loop structure.

Let M be a non empty multiplicative loop structure. The functor units(M) yielding a subset of M is defined by the term

(Def. 8) $\{x$, where x is an element of $M : x$ is unital$\}$.

Now we state the proposition:

(93) Let us consider a right unital, non empty multiplicative loop structure M, and an element e_2 of M. Then e_2 is unital if and only if $e_2 \in$ units(M).

Let us consider M. Let u, v be unital elements of M. Let us observe that $u \cdot v$ is unital and units(M) is (the multiplication of M)-binopclosed.

The functor UnitGroup(M) yielding a strict multiplicative magma is defined by the term

(Def. 9) $\langle$units(M), (the multiplication of M) $\restriction$ units$(M)\rangle$.

Now we state the proposition:

(94) $1_M \in$ UnitGroup(M).

Let us consider M. Let us observe that UnitGroup(M) is non empty.

Now we state the proposition:

(95) Let us consider elements a, b of M, and elements x, y of UnitGroup(M). If $a = x$ and $b = y$, then $a \cdot b = x \cdot y$.

Let us consider M. One can check that UnitGroup(M) is associative.

Let M be an associative, left, right unital, non empty multiplicative loop structure. Let us observe that UnitGroup(M) is group-like.

Now we state the proposition:

(96) Let us consider an associative, well unital, non empty multiplicative loop structure M. Then $\mathbf{1}_{\mathrm{UnitGroup}(M)} = 1_M$. The theorem is a consequence of (95).

Let R be a commutative ring. Let us note that UnitGroup(R) is commutative.

Now we state the proposition:

(97) Let us consider an associative, left, right unital, non empty multiplicative loop structure M. Then UnitGroup(M) is a strict group.

Let us consider an associative, left, right unital, non empty multiplicative loop structure G. Now we state the propositions:

(98) If G is group-like, then units(G) = the carrier of G.

(99) If G is group-like, then UnitGroup(G) = the multiplicative magma of G. The theorem is a consequence of (98).

Let us consider a skew field R. Now we state the propositions:

(100) NonZero R = units(R).

Proof: For every object x such that $x \in$ NonZero R holds $x \in$ units(R) by [16, (7)]. For every object x such that $x \in$ units(R) holds $x \in$ NonZero R. □

(101) MultGroup(R) = UnitGroup(R). The theorem is a consequence of (100).

Let R be a skew field. We identify NonZero R with units(R). We identify MultGroup(R) with UnitGroup(R). Now we state the propositions:

(102) Let us consider a commutative ring R, and an object x. Then $x \in \mathrm{Z}(R)$ and $x \in$ UnitGroup(R) if and only if $x \in \mathrm{Z}(\mathrm{UnitGroup}(R))$. The theorem is a consequence of (6).

(103) Let us consider an object x. Suppose $x \in \mathrm{Z}(R)$ and $x \in$ UnitGroup(R). Then $x \in \mathrm{Z}(\mathrm{UnitGroup}(R))$.

Proof: Reconsider $a = x$ as an element of UnitGroup(R). For every element g of UnitGroup(R), $a \cdot g = g \cdot a$ by (95), [3, (43)]. □

Let us consider a square matrix A over R of dimension n and an element z of R. Now we state the propositions:

(104) If $A = z \cdot (I_R^{n \times n})$ and $z \in$ UnitGroup(R), then A is invertible. The theorem is a consequence of (93), (89), (88), and (22).

(105) If $n > 0$ and $A = z \cdot (I_R^{n \times n})$ and A is invertible, then $z \in$ UnitGroup(R).

Proof: Consider M_2 being a square matrix over R of dimension n such that A is inverse of M_2. $I_R^{n\times n} = z \cdot (I_R^{n\times n} \cdot M_2)$. $\langle 1, 1\rangle \in$ the indices of M_2 and $\langle 1, 1\rangle \in$ the indices of $I_R^{n\times n}$ by [4, (19)]. $I_R^{n\times n} = M_2 \cdot z \cdot (I_R^{n\times n})$. $\square$

Now we state the propositions:

(106) Let us consider an invertible square matrix A over R of dimension n. Suppose for every invertible square matrix B over R of dimension n, $A \cdot B = B \cdot A$. Then there exists an element z of R such that

(i) $z \in \mathrm{Z}(\mathrm{UnitGroup}(R))$, and

(ii) $A = z \cdot (I_R^{n\times n})$.

Proof: For every natural numbers i, j such that $i \neq j$ holds $A \cdot (\mathrm{canMat}(i, j, n, R)) = (\mathrm{canMat}(i, j, n, R)) \cdot A$. For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of A and $i \neq j$ holds $A_{i,j} = 0_R$ by [12, (87)], (43), (42). For every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of A holds $A_{i,i} = A_{j,j}$. $\square$

(107) Let us consider rings R_1, R_2. Suppose R_1 and R_2 are isomorphic. Then UnitGroup(R_1) and UnitGroup(R_2) are isomorphic.

Proof: Consider φ being a function from R_1 into R_2 such that φ inherits ring isomorphism. Set $\psi = \varphi\restriction$(the carrier of UnitGroup(R_1)). For every element a of UnitGroup(R_1), there exists an element x of R_1 such that $a = x$ and $\varphi(x) = \psi(a)$ by [10, (49)]. For every elements a, b of UnitGroup(R_1), $\psi(a \cdot b) = \psi(a) \cdot \psi(b)$ by (93), [12, (87)], [10, (49)], (95). For every object y such that $y \in$ the carrier of UnitGroup(R_2) there exists an object x such that $x \in$ the carrier of UnitGroup(R_1) and $y = \psi(x)$. $\square$

11. General Linear Group over a Ring

Let us consider n and R. The functor $\mathrm{GL}_n(R)$ yielding a strict group is defined by the term

(Def. 10) UnitGroup(Matrix-ring(n, R)).

One can verify that $\mathrm{GL}_0(R)$ is trivial.

Now we state the proposition:

(108) Let us consider an object x. Then x is an element of $\mathrm{GL}_n(R)$ if and only if x is an invertible square matrix over R of dimension n. The theorem is a consequence of (93), (60), and (63).

Let us consider R and n. One can verify that $\mathrm{GL}_n(R)$ is (n,R)-matrix-membered and the carrier of $\mathrm{GL}_n(R)$ is non empty and the carrier of $\mathrm{GL}_n(R)$ is (n,R)-matrix-membered.

Now we state the proposition:

(109) Let us consider elements x, y of $\mathrm{GL}_n(R)$, and square matrices X, Y over R of dimension n. If $X = x$ and $Y = y$, then $X \cdot Y = x \cdot y$. The theorem is a consequence of (60), (95), and (63).

The scheme *GLSubset* deals with a natural number n and a ring $\mathcal{R}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists a subset A of $\mathrm{GL}_n(\mathcal{R})$ such that for every invertible square matrix X over $\mathcal{R}$ of dimension n, $X \in A$ iff $\mathcal{P}[X]$.

The scheme *GLSubset2* deals with a natural number n and a ring $\mathcal{R}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 2) There exists a subset A of $\mathrm{GL}_n(\mathcal{R})$ such that for every square matrix X over $\mathcal{R}$ of dimension n such that X is invertible holds $X \in A$ iff $\mathcal{P}[X]$.

Now we state the propositions:

(110) Let us consider a square matrix X over R of dimension n. Then X is an element of $\mathrm{GL}_n(R)$ if and only if X is invertible.

(111) $I_R^{n \times n}$ is an element of $\mathrm{GL}_n(R)$.

(112) $\mathbf{1}_{\mathrm{GL}_n(R)} = I_R^{n \times n}$.

Proof: Reconsider $I = I_R^{n \times n}$ as an element of $\mathrm{GL}_n(R)$. For every element x of $\mathrm{GL}_n(R)$, $x \cdot I = x$ and $I \cdot x = x$ by (108), (109), [24, (19), (18)]. $\square$

(113) Let us consider an element x of $\mathrm{GL}_n(R)$, and an invertible square matrix X over R of dimension n. If $x = X$, then $x^{-1} = X^{\smile}$. The theorem is a consequence of (108), (112), and (109).

(114) Let us consider an element g of $\mathrm{GL}_n(R)$, and square matrices X, Y over R of dimension n. If $X = g$, then $Y = \cdot^{-1}_{\mathrm{GL}_n(R)}(g)$ iff Y is inverse of X. The theorem is a consequence of (109), (112), and (110).

The scheme *GLSubgroup* deals with a natural number n and a ring $\mathcal{R}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 3) There exists a strict subgroup H of $\mathrm{GL}_n(\mathcal{R})$ such that for every invertible square matrix X over $\mathcal{R}$ of dimension n, $X \in H$ iff $\mathcal{P}[X]$

provided

- for every invertible square matrices X, Y over $\mathcal{R}$ of dimension n such that $\mathcal{P}[X]$ and $\mathcal{P}[Y]$ holds $\mathcal{P}[X \cdot Y]$ and
- for every invertible square matrix X over $\mathcal{R}$ of dimension n such that $\mathcal{P}[X]$ holds $\mathcal{P}[X^{\smile}]$ and
- $\mathcal{P}[I_{\mathcal{R}}^{n \times n}]$.

The scheme *GLSubgroup2* deals with a natural number n and a ring $\mathcal{R}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 4) There exists a strict subgroup H of $\mathrm{GL}_n(\mathcal{R})$ such that for every invertible square matrix X over $\mathcal{R}$ of dimension n, $X \in H$ iff $\mathcal{P}[X]$

provided

- for every invertible square matrices X, Y over $\mathcal{R}$ of dimension n such that $\mathcal{P}[X]$ and $\mathcal{P}[Y]$ holds $\mathcal{P}[X \cdot Y]$ and
- for every invertible square matrix X over $\mathcal{R}$ of dimension n such that $\mathcal{P}[X]$ holds $\mathcal{P}[X^{\smile}]$ and
- $\mathcal{P}[I_{\mathcal{R}}^{n \times n}]$.

12. Center of General Linear Group

Now we state the propositions:

(115) Let us consider an element X of $\mathrm{GL}_n(R)$. Suppose there exists an element z of R such that $z \in \mathrm{Z}(R)$ and $X = z \cdot (I_R^{n \times n})$. Then $X \in \mathrm{Z}(\mathrm{GL}_n(R))$. The theorem is a consequence of (108), (109), and (87).

(116) Let us consider an invertible square matrix X over R of dimension n. Suppose $n \neq 1$. Then $X \in \mathrm{Z}(\mathrm{GL}_n(R))$ if and only if there exists an element z of R such that $z \in \mathrm{Z}(R)$ and $z \in \mathrm{UnitGroup}(R)$ and $X = z \cdot (I_R^{n \times n})$. The theorem is a consequence of (108), (109), and (87).

(117) Let us consider an element r of R. Suppose $r \in \mathrm{Z}(\mathrm{UnitGroup}(R))$. Let us consider an element s of R. If $s \in \mathrm{UnitGroup}(R)$, then $r \cdot s = s \cdot r$. The theorem is a consequence of (95).

(118) Let us consider an invertible square matrix X over R of dimension n. Suppose $n = 1$. Then $X \in \mathrm{Z}(\mathrm{GL}_n(R))$ if and only if there exists an element z of R such that $z \in \mathrm{Z}(\mathrm{UnitGroup}(R))$ and $X = z \cdot (I_R^{n \times n})$. The theorem is a consequence of (108), (109), (28), (105), (89), (88), and (117).

(119) $\mathrm{Z}(\mathrm{GL}_1(R))$ and $\mathrm{Z}(\mathrm{UnitGroup}(R))$ are isomorphic. The theorem is a consequence of (67), (107), and (5).

(120) Let us consider a commutative ring R, and an invertible square matrix X over R of dimension n. Then $X \in \mathrm{Z}(\mathrm{GL}_n(R))$ if and only if there exists an element z of R such that $z \in \mathrm{Z}(R)$ and $z \in \mathrm{UnitGroup}(R)$ and $X = z \cdot (I_R^{n \times n})$. The theorem is a consequence of (118), (102), and (116).

13. Extension of Scalars for the General Linear Group

Now we state the propositions:

(121) Let us consider a square matrix M_1 over R of dimension n, and a square matrix M_2 over S of dimension n. If $M_1 = M_2$ and M_2 is invertible, then M_1 is invertible. The theorem is a consequence of (70), (79), and (82).

(122) The carrier of $\mathrm{GL}_n(S) \subseteq$ the carrier of $\mathrm{GL}_n(R)$. The theorem is a consequence of (108), (70), (110), and (121).

(123) The multiplication of $\mathrm{GL}_n(S) =$ (the multiplication of $\mathrm{GL}_n(R)$) $\restriction$ (the carrier of $\mathrm{GL}_n(S)$).
Proof: Set $U_6 =$ the carrier of $\mathrm{GL}_n(S)$. Set $U_5 =$ the carrier of $\mathrm{GL}_n(R)$. Set $f_2 =$ the multiplication of $\mathrm{GL}_n(S)$. Set $f_1 =$ the multiplication of $\mathrm{GL}_n(R)$. $U_6 \subseteq U_5$. For every elements x, y of U_6, $f_2(x, y) = (f_1 \restriction U_6)(x, y)$ by [11, (5)], (108), (109), (79). □

(124) $\mathrm{GL}_n(S)$ is a strict subgroup of $\mathrm{GL}_n(R)$. The theorem is a consequence of (122) and (123).

14. Appendix 1: Matrix Inversion as a Unary Operator

Now we state the proposition:

(125) Let us consider an element x of Matrix-ring(n, R). Then x is invertible if and only if there exists an invertible square matrix X over R of dimension n such that $X = x$. The theorem is a consequence of (60) and (63).

Let us consider n and R. Note that there exists an element of Matrix-ring(n, R) which is invertible.

Now we state the proposition:

(126) Let us consider an object x. Then x is an invertible element of Matrix-ring(n, R) if and only if x is an invertible square matrix over R of dimension n. The theorem is a consequence of (125) and (60).

Let us consider n and R. The functor n-invertible-Matrices-over R yielding a subset of $R^{n \times n}$ is defined by the term

(Def. 11) $\{X$, where X is an element of $R^{n \times n}$: X is an invertible square matrix over R of dimension $n\}$.

Now we state the proposition:

(127) Let us consider an object x. Then $x \in n$-invertible-Matrices-over R if and only if x is an invertible square matrix over R of dimension n.

Let us consider n and R. One can verify that n-invertible-Matrices-over R is non empty.

Now we state the propositions:

(128) The carrier of $\mathrm{GL}_n(R) = n$-invertible-Matrices-over R. The theorem is a consequence of (108) and (127).

(129) n-invertible-Matrices-over R is (matrix-mult(n, R))-binopclosed.
PROOF: Set $U = n$-invertible-Matrices-over R. For every set x such that $x \in U \times U$ holds (matrix-mult$(n, R))(x) \in U$ by (127), [22, (36)]. □

(130) One(R, n) is a unity w.r.t. matrix-mult(n, R).
PROOF: Set $\mu =$ matrix-mult(n, R). For every element x of $R^{n \times n}$, $\mu(\mathrm{One}(R, n), x) = x$ and $\mu(x, \mathrm{One}(R, n)) = x$ by [24, (18), (19)]. □

Let us consider n and R. Note that matrix-mult(n, R) is associative and matrix-mult(n, R) is unital.

Now we state the proposition:

(131) $\mathrm{One}(R, n) = \mathbf{1}_{\text{matrix-mult}(n,R)}$. The theorem is a consequence of (130).

Let us consider n and R. The functor matrix-inv(n, R) yielding a unary operation on n-invertible-Matrices-over R is defined by

(Def. 12) for every element x of n-invertible-Matrices-over R and for every invertible square matrix X over R of dimension n such that $x = X$ holds $it(x) = X^{\smile}$.

Now we state the proposition:

(132) $\cdot^{-1}_{\mathrm{GL}_n(R)} =$ matrix-inv(n, R). The theorem is a consequence of (128), (108), and (113).

Let us consider a binary operation F on n-invertible-Matrices-over R. Now we state the propositions:

(133) If $F =$ matrix-mult$(n, R) \restriction (n$-invertible-Matrices-over $R)$, then One(R, n) is a unity w.r.t. F.
PROOF: Reconsider $e_1 = \mathrm{One}(R, n)$ as an element of n-invertible-Matrices-over R. For every element x of n-invertible-Matrices-over R, $F(e_1, x) = x$ and $F(x, e_1) = x$ by (127), [10, (49)], [12, (87)], [24, (18), (19)]. □

(134) If $F =$ matrix-mult$(n, R) \restriction (n$-invertible-Matrices-over $R)$, then $\mathrm{One}(R, n) = \mathbf{1}_F$. The theorem is a consequence of (127) and (133).

(135) Suppose $F =$ matrix-mult$(n, R) \restriction (n$-invertible-Matrices-over $R)$. Then matrix-inv(n, R) is an inverse operation w.r.t. F.
PROOF: Set $i =$ matrix-inv(n, R). Reconsider $e_1 = \mathrm{One}(R, n)$ as an element of n-invertible-Matrices-over R. $e_1 = \mathbf{1}_F$. For every element x of n-invertible-Matrices-over R, $F(x, i(x)) = \mathbf{1}_F$ and $F(i(x), x) = \mathbf{1}_F$ by (127), [10, (49)]. □

(136) If $F =$ matrix-mult$(n, R) \restriction (n$-invertible-Matrices-over $R)$, then F is unital and associative and has inverse operation.

PROOF: For every elements a, b, c of n-invertible-Matrices-over R, $F(a, F(b, c)) = F(F(a, b), c)$ by (127), [10, (49)], [24, (33)]. □

(137) Suppose $F = \text{matrix-mult}(n, R) \restriction (n\text{-invertible-Matrices-over } R)$. Then $\text{matrix-inv}(n, R) =$ the inverse operation w.r.t. F.
PROOF: F is unital and associative and has inverse operation. For every element x of n-invertible-Matrices-over R, $(\text{matrix-inv}(n, R))(x) =$ (the inverse oper $F)(x)$ by (127), (134), [8, (59)], [10, (49)]. □

Let us consider elements a, b of R. Now we state the propositions:

(138) RUDIN'S RESULTS [17] CONCERNING TWO-SIDED INVERSES AND UNIQUE RIGHT INVERTIBILITY, AS DISCUSSED BY ALMIRA AND CID [1]:
If $a \cdot b = 1_R$, then for every element b' of R such that $a \cdot b' = 1_R$ holds $b' = b$ iff for every element x of R, $a \cdot x \neq 0_R$ or $x = 0_R$.

(139) If $a \cdot b = 1_R$, then for every element x of R, $a \cdot x \neq 0_R$ or $x = 0_R$ iff a is invertible.

(140) If $a \cdot b = 1_R$, then for every element b' of R such that $a \cdot b' = 1_R$ holds $b' = b$ iff a is invertible.

15. APPENDIX 2: *Linear* GROUPS

Now we state the proposition:

(141) Let us consider a group G. Then the multiplicative magma of G is a subgroup of G.

Let G be a group. Let us note that the multiplicative magma of G is associative and group-like.

Now we state the proposition:

(142) Let us consider a group G. Then $\cdot_G^{-1} = \cdot_\alpha^{-1}$, where α is the multiplicative magma of G. The theorem is a consequence of (141).

Let n be a natural number, R be a ring, and I_1 be a group. We say that I_1 is (n,R)-linear if and only if

(Def. 13) the multiplicative magma of I_1 is a subgroup of $\mathrm{GL}_n(R)$.

Let us consider n and R. Let us observe that $\mathrm{GL}_n(R)$ is (n,R)-linear and there exists a group which is strict and (n,R)-linear.

From now on G denotes an (n,R)-linear group.

Now we state the proposition:

(143) Every subgroup of $\mathrm{GL}_n(R)$ is (n,R)-linear. The theorem is a consequence of (141).

Let us consider n and R. Observe that every subgroup of $\mathrm{GL}_n(R)$ is (n,R)-linear.

Now we state the propositions:

(144) The carrier of $G \subseteq$ the carrier of $\mathrm{GL}_n(R)$.

(145) The carrier of $G \subseteq n$-invertible-Matrices-over R. The theorem is a consequence of (128) and (144).

Let us consider n and R. Let us note that every group which is (n,R)-linear is also (n,R)-matrix-membered.

Now we state the propositions:

(146) $\mathbf{1}_G = \mathbf{1}_{\mathrm{GL}_n(R)}$.

(147) $\mathbf{1}_G = I_R^{n\times n}$. The theorem is a consequence of (146) and (112).

Let us consider n, R, and G. One can verify that the carrier of G is non empty and (n,R)-matrix-membered.

Now we state the propositions:

(148) The multiplication of $G =$ (the multiplication of $\mathrm{GL}_n(R)$) $\restriction$ (the carrier of G).

(149) The multiplication of $G =$ matrix-mult$(n, R) \restriction$ (the carrier of G). The theorem is a consequence of (144).

(150) The multiplication of $\mathrm{GL}_n(R) =$ matrix-mult$(n, R) \restriction$ (n-invertible-Matrices-over R

(151) Every subgroup of G is (n,R)-linear.

Proof: The carrier of $H \subseteq$ the carrier of $\mathrm{GL}_n(R)$. The multiplication of $H =$ (the multiplication of $\mathrm{GL}_n(R)$) $\restriction$ (the carrier of H) by [12, (96)], (148), [10, (51)]. □

Let us consider n, R, and G. Note that every subgroup of G is (n,R)-linear.

Now we state the propositions:

(152) $\cdot_G^{-1} = \cdot_{\mathrm{GL}_n(R)}^{-1} \restriction$(the carrier of G). The theorem is a consequence of (142).

(153) $\cdot_G^{-1} =$ matrix-inv$(n, R) \restriction$(the carrier of G). The theorem is a consequence of (152) and (132).

16. Appendix 3: *General* Linear Groups

Let us consider n and R. Let I_1 be an (n,R)-linear group. We say that I_1 is general if and only if

(Def. 14) for every object x, $x \in I_1$ iff x is an invertible square matrix over R of dimension n.

Observe that $\mathrm{GL}_n(R)$ is general and there exists an (n,R)-linear group which is general and strict and there exists an (n,R)-linear group which is general.

Let us consider a general, (n,R)-linear group G. Now we state the propositions:

(154) The carrier of G = the carrier of $\mathrm{GL}_n(R)$. The theorem is a consequence of (108).

(155) The multiplication of G = the multiplication of $\mathrm{GL}_n(R)$. The theorem is a consequence of (154).

(156) The multiplicative magma of $G = \mathrm{GL}_n(R)$. The theorem is a consequence of (155) and (154).

Now we state the proposition:

(157) The general , (n,R)-linear , strict group = $\mathrm{GL}_n(R)$.

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References

[1] J.M. Almira and J. Ángel Cid. A ring-theoretic approach to the double-sidedness of the matrix inverse. *Elemente der Mathematik*, 79(1):30–33, 2023. doi:10.4171/EM/496.

[2] Broderick Arneson and Piotr Rudnicki. Primitive roots of unity and cyclotomic polynomials. *Formalized Mathematics*, 12(**1**):59–67, 2004.

[3] Broderick Arneson, Matthias Baaz, and Piotr Rudnicki. Witt's proof of the Wedderburn theorem. *Formalized Mathematics*, 12(**1**):69–75, 2004.

[4] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[6] Nicolas Bourbaki. *Elements of Mathematics. Algebra I. Chapters 1-3.* Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1989.

[7] Nicolas Bourbaki. *Algebra: Chapter 8.* Springer Nature, 2023.

[8] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[9] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[10] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[11] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[13] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2 (**4**):475–480, 1991.

[14] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[15] Tsit-Yuen Lam. *A first course in noncommutative rings*, volume 131. Springer, 1991.

[16] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[17] Walter Rudin. Unique right inverses are two-sided. *The American Mathematical Monthly*, 92(7):489–490, 1985. doi:10.1080/00029890.1985.11971659.

[18] Christoph Schwarzweller. Ring and field adjunctions, algebraic elements and minimal polynomials. *Formalized Mathematics*, 28(**3**):251–261, 2020. doi:10.2478/forma-2020-0022.

[19] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(**4**):461–466, 1991.

[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[21] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.

[22] Xiaopeng Yue, Xiquan Liang, and Zhongpin Sun. Some properties of some special matrices. *Formalized Mathematics*, 13(**4**):541–547, 2005.

[23] Katarzyna Zawadzka. The sum and product of finite sequences of elements of a field. *Formalized Mathematics*, 3(**2**):205–211, 1992.

[24] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.

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