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Scaling Properties of Jordan, Borel and Lebesgue Measures under Affine Maps[1]

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Summary. In this article we formalize the classical scaling formula for measures under affine transformations of the real line. We begin by proving that affine maps send intervals to intervals and scale their Jordan measure by the absolute value of the linear coefficient. The result is then extended to the field generated by intervals through a collection of auxiliary lemmas concerning images of set families, additive set functions and sums of extended real-valued sequences.

We further show that every affine map with non-zero linear coefficient preserves the Borel σ-field, which yields the corresponding scaling property for Borel measure. Finally, we establish the invariance of Lebesgue measurability under affine transformations and prove that Lebesgue measure satisfies

$$L - Meas(aA + b) = |a|\, L - Meas(A).$$

These results provide a unified formalization of the scaling behavior of Jordan, Borel and Lebesgue measures under affine maps.

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Introduction

The scaling properties of measures under affine transformations constitute a classical topic in measure theory. In particular, if $T(x) = ax + b$ is an affine map of the real line and λ denotes the Lebesgue measure, then for every Lebesgue measurable set A,

$$\lambda(T(A)) = |a|\,\lambda(A).$$

This scaling property, together with the invariance of measurable structures under affine transformations, is a standard result appearing in measure-theoretic textbooks such as Halmos [13] and Folland [11].

The purpose of the present article is to formalize these results in the Mizar system [3], [4]. We start with intervals and prove that affine maps preserve intervals and scale their Jordan measure by the absolute value of the linear coefficient. To extend this result, we establish several auxiliary lemmas concerning images of finite families of sets, additive set functions, and sums of extended real-valued sequences.

Next, we investigate the action of affine maps on generated set families. We show that every affine map with non-zero linear coefficient preserves the family of intervals generating the Borel σ-field and consequently preserves the Borel σ-field itself. This allows us to derive the corresponding scaling formula for Borel measure.

Finally, we prove that affine maps preserve Lebesgue measurability and establish the scaling property of Lebesgue measure,

$$L - Meas(aA + b) = |a|\,L - Meas(A).$$

As a consequence, the article provides a unified formalization of the scaling behavior of Jordan, Borel, and Lebesgue measures under affine transformations of the real line.

1. Affine Maps on Intervals and Jordan Measure

Now we state the propositions:

(1) Let us consider real numbers a, b, and an interval I. Then $(\mathrm{AffineMap}(a, b))^\circ I$ is an interval.

(2) Let us consider real numbers a, b, and an interval subset I of $\mathbb{R}$. Then $(\text{J-Meas})((\mathrm{AffineMap}(a, b))^\circ I) = |a| \cdot (\text{J-Meas})(I)$. The theorem is a consequence of (1).

(3) Let us consider non empty sets X, Y, a function f from X into Y, and a function F. Suppose $\operatorname{rng} F \subseteq 2^X$. Then

(i) $\operatorname{dom}((^\circ f) \cdot F) = \operatorname{dom} F$, and

(ii) for every object i such that $i \in \operatorname{dom}((^\circ f) \cdot F)$ holds $((^\circ f) \cdot F)(i) = f^\circ F(i)$.

(4) Let us consider real numbers a, b, and a finite sequence F of elements of $\mathrm{Intervals}_{\mathbb{R}}$. Then

(i) $(^\circ(\mathrm{AffineMap}(a,b))) \cdot F$ is a finite sequence of elements of $\mathrm{Intervals}_{\mathbb{R}}$, and

(ii) for every natural number i such that $i \in \operatorname{dom}((^\circ(\mathrm{AffineMap}(a,b))) \cdot F)$ holds $((^\circ(\mathrm{AffineMap}(a,b))) \cdot F)(i) = (\mathrm{AffineMap}(a,b))^\circ F(i)$.

PROOF: Set $G = (^\circ(\mathrm{AffineMap}(a,b))) \cdot F$. Reconsider $F_1 = F$ as a finite sequence of elements of $2^{\mathbb{R}}$. For every natural number i such that $i \in \operatorname{dom}((^\circ(\mathrm{AffineMap}(a,b))) \cdot F_1)$ holds $((^\circ(\mathrm{AffineMap}(a,b))) \cdot F_1)(i) = (\mathrm{AffineMap}(a,b))^\circ F_1(i)$. For every object z such that $z \in \operatorname{rng} G$ holds $z \in \mathrm{Intervals}_{\mathbb{R}}$ by [8, (3)], (3), (1). □

(5) Let us consider non empty sets X, Y, a function f from X into Y, and a finite sequence F. Suppose $\operatorname{rng} F \subseteq 2^X$ and f is one-to-one and F is disjoint valued. Then $(^\circ f) \cdot F$ is disjoint valued.
PROOF: Set $G = (^\circ f) \cdot F$. For every objects x, y such that $x \neq y$ holds $G(x)$ misses $G(y)$ by (3), [5, (62)]. □

(6) Let us consider non empty sets X, Y, a function f from X into Y, and a function F. If $\operatorname{rng} F \subseteq 2^X$, then $\bigcup(^\circ f) \cdot F = f^\circ(\bigcup F)$.

(7) Let us consider a set X, a non empty family S of subsets of X, a function M from S into $\overline{\mathbb{R}}$, and a finite sequence F of elements of S. Then there exists a finite sequence x of elements of $\overline{\mathbb{R}}$ such that

(i) $\operatorname{dom} x = \operatorname{dom} F$, and

(ii) for every natural number i such that $i \in \operatorname{dom} x$ holds $x(i) = M(F(i))$.

(8) Let us consider a set X, a non empty family S of subsets of X, a function M from S into $\overline{\mathbb{R}}$, a disjoint valued finite sequence F of elements of S, and a finite sequence x of elements of $\overline{\mathbb{R}}$. Suppose $\emptyset \in S$ and S is $\cup$-closed and M is zeroed and additive and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} x$ holds $x(i) = M(F(i))$. Then $\sum x = M(\bigcup F)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every disjoint valued finite sequence F of elements of S for every finite sequence x of elements of $\overline{\mathbb{R}}$ such that $\operatorname{len} F = \$_1$ and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} x$ holds $x(i) = M(F(i))$ holds $\sum x = M(\bigcup F)$. $\mathcal{P}[0]$ by [9, (2)], [10, (7)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [12, (6)], [14, (59)], [1, (11)], [2, (59), (5)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(9) Let us consider a real number r, and finite sequences x, y of elements of $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} x = \operatorname{dom} y$ and for every natural number i such that $i \in \operatorname{dom} x$ holds $y(i) = r \cdot x(i)$. Then $\sum y = r \cdot (\sum x)$.

(10) $\langle \emptyset \rangle$ is a disjoint valued finite sequence of elements of $\mathrm{Intervals}_{\mathbb{R}}$.

(11) Let us consider real numbers a, b, and an element I of the field generated by $\mathrm{Intervals}_{\mathbb{R}}$. Then $(\mathrm{AffineMap}(a,b))^\circ I \in$ the field generated by $\mathrm{Intervals}_{\mathbb{R}}$. The theorem is a consequence of (4) and (6).

(12) Let us consider an element I of the field generated by $\mathrm{Intervals}_{\mathbb{R}}$. Then

(i) $(\text{J-Meas})(I) = (\text{B-Meas})(I)$, and

(ii) $(\text{J-Meas})(I) = (\text{L-Meas})(I)$.

(13) Let us consider real numbers a, b, and an element I of the field generated by $\mathrm{Intervals}_{\mathbb{R}}$. Then $(\text{J-Meas})((\mathrm{AffineMap}(a,b))^\circ I) = |a| \cdot (\text{J-Meas})(I)$. The theorem is a consequence of (4), (3), (5), (6), (7), (8), (2), and (9).

2. Affine Invariance of Borel Structures

Now we state the propositions:

(14) Let us consider a set X. Then ${}^\circ\mathrm{id}_X = \mathrm{id}_{2^X}$.
PROOF: For every object A such that $A \in 2^X$ holds $({}^\circ\mathrm{id}_X)(A) = A$ by [5, (92)]. □

(15) Let us consider non empty sets X, Y, a function F from X into Y, and a non empty set S. Suppose $S \subseteq 2^X$ and F is bijective. Then S is a σ-field of subsets of X if and only if $({}^\circ F)^\circ S$ is a σ-field of subsets of Y. The theorem is a consequence of (14).

(16) Let us consider non empty sets X, Y, a function F from X into Y, and a family S of subsets of X. Suppose F is bijective. Then $({}^\circ F)^\circ(\sigma(S)) = \sigma(({}^\circ F)^\circ S)$.
PROOF: Reconsider $G = ({}^\circ F)^\circ(\sigma(S))$ as a σ-field of subsets of Y. Set $K = ({}^\circ F)^\circ S$. For every set Z such that $K \subseteq Z$ and Z is a σ-field of subsets of Y holds $G \subseteq Z$ by [6, (25)], [5, (33)], [14, (126)], [7, (18)]. □

Let us consider subsets A, B of $\overline{\mathbb{R}}$ and a real number a. Now we state the propositions:

(17) Suppose $0 < a$ and $B = \{a \cdot r$, where r is an extended real number : $r \in A\}$. Then

(i) $\sup B = a \cdot (\sup A)$, and

(ii) $\inf B = a \cdot (\inf A)$.

(18) Suppose $a < 0$ and $B = \{a \cdot r$, where r is an extended real number : $r \in A\}$. Then

(i) $\inf B = a \cdot (\sup A)$, and

(ii) $\sup B = a \cdot (\inf A)$.

Now we state the proposition:

(19) Let us consider sequences p, q of $\overline{\mathbb{R}}$, and a real number a. Suppose $0 \leqslant a$ and for every element n of $\mathbb{N}$, $q(n) = a \cdot p(n)$. Then $\overline{\sum} q = a \cdot (\overline{\sum} p)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv (\operatorname{Ser} q)(\$_1) = a \cdot (\operatorname{Ser} p)(\$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. For every object z, $z \in \operatorname{rng} \operatorname{Ser} q$ iff $z \in \{a \cdot r$, where r is an extended real number : $r \in \operatorname{rng} \operatorname{Ser} p\}$ by [6, (4), (113)]. □

Let us consider real numbers a, b. Now we state the propositions:

(20) $(^\circ(\text{AffineMap}(a,b)))^\circ(\text{Intervals}_{\mathbb{R}}) \subseteq \text{Intervals}_{\mathbb{R}}$.
PROOF: For every object x such that $x \in (^\circ(\text{AffineMap}(a,b)))^\circ(\text{Intervals}_{\mathbb{R}})$ holds $x \in \text{Intervals}_{\mathbb{R}}$ by [7, (7)], (1). □

(21) If $a \neq 0$, then $(^\circ(\text{AffineMap}(a,b)))^\circ(\text{Intervals}_{\mathbb{R}}) = \text{Intervals}_{\mathbb{R}}$.
PROOF: Set $f = \text{AffineMap}(a,b)$. For every object z such that $z \in \text{Intervals}_{\mathbb{R}}$ holds $z \in (^\circ f)^\circ(\text{Intervals}_{\mathbb{R}})$ by (1), [7, (7)], [14, (126)], [5, (92)]. □

(22) If $a \neq 0$, then $(^\circ(\text{AffineMap}(a,b)))^\circ(\text{the Borel sets}) = \text{the Borel sets}$. The theorem is a consequence of (16) and (21).

Now we state the proposition:

(23) Let us consider real numbers a, b, and an element I of the Borel sets. Then $(\text{AffineMap}(a,b))^\circ I \in$ the Borel sets. The theorem is a consequence of (22).

3. SCALING PROPERTIES OF BOREL AND LEBESGUE MEASURES

Now we state the proposition:

(24) Let us consider real numbers a, b, and an element I of the Borel sets. Then $\text{(B-Meas)}((\text{AffineMap}(a,b))^\circ I) = |a| \cdot \text{(B-Meas)}(I)$. The theorem is a consequence of (23), (11), (6), (13), (19), and (17).

Let us consider real numbers a, b and an element I of L-Field. Now we state the propositions:

(25) $(\text{AffineMap}(a,b))^\circ I \in$ L-Field. The theorem is a consequence of (23) and (24).

(26) $\text{(L-Meas)}((\text{AffineMap}(a,b))^\circ I) = |a| \cdot \text{(L-Meas)}(I)$. The theorem is a consequence of (24) and (23).

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