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Rational Numbers, Irrational Roots and Numeral Representations

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Summary. The formalized representation of numbers in positional systems in [6] can be difficult in some applications as it is based on zero-based finite sequences, as introduced by [12], which are slightly less developed in Mizar despite their obvious benefits [9]. In the first part of the paper, a number of tools have been proposed to facilitate the practical use of decimal notation, especially in the context of divisibility. Some theorems related to rounding real numbers to integers, as shown in [11] have also been supplemented. Though the study aim to expand current Mizar Mathematical Liberary, the problems with its size and usability are hopefully could probably be solved with the new tools such as recently introduced MMLKG [10].

The second part of the article formalizes an obvious corollary arising from [13] that all non integer roots of natural numbers are irrational. As the notion of irrationality of some square roots dates back to Theodorus (465–398 BC) [2] it should be considered the introductory mathematics. Therefore it seems that the relevant theorems are well fitted to Mizar Mathematical Library [7].

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1. Preliminaries

Let a be a heavy, positive real number and n be a non zero natural number. Let us note that $\sqrt[n]{a}$ is heavy and positive.

Let a be an integer and n be a natural number. One can verify that $a^{n+1} \operatorname{div} a^n$ reduces to a.

Let a be an even natural number. Note that $\frac{a}{2} \bmod a$ reduces to $\frac{a}{2}$.

Let n be a non trivial natural number. Observe that $a^n \bmod 2 \cdot a$ is zero.

Let a be an odd natural number. Note that $\frac{a-1}{2} \bmod a$ reduces to $\frac{a-1}{2}$. Let n be a non zero natural number. Observe that $a^n \bmod 2 \cdot a$ reduces to a.

Let a be a natural number and b be a non trivial natural number. One can check that $a \bmod a \cdot b$ reduces to a.

Observe that $a \operatorname{div} a \cdot b$ is zero.

Let b be a natural number and c be a non zero natural number. Let us note that $(a \bmod b) \bmod b \cdot c$ reduces to $a \bmod b$.

Let a be a non trivial, odd natural number. One can verify that $\frac{a+1}{2} \bmod a$ reduces to $\frac{a+1}{2}$.

Now we state the propositions:

(1) Let us consider integers a, b, and a natural number n. If $a \leqslant b \leqslant a + n$, then $b = a + 0$ or ... or $b = a + n$.

(2) Let us consider integers a, b. Then $a \leqslant b \leqslant a + 1$ if and only if $b = a$ or $b = a + 1$. The theorem is a consequence of (1).

Let a be a non integer real number. Let us note that $\operatorname{sgn}(\operatorname{frac} a)$ is weightless and positive.

One can verify that $a \cdot (\operatorname{sgn}(\operatorname{frac} a))$ reduces to a.

Let a be a light, non negative real number. Note that $\operatorname{frac} a$ reduces to a.

Let us consider a real number r. Now we state the propositions:

(3) (i) $r \cdot (\operatorname{sgn}(r)) = |r|$, and

(ii) $\frac{r}{\operatorname{sgn}(r)} = |r|$, and

(iii) $r = \frac{|r|}{\operatorname{sgn}(r)}$.

(4) $\lfloor |r| \rfloor + \operatorname{frac} |r| = r \cdot (\operatorname{sgn}(r))$. The theorem is a consequence of (3).

Now we state the proposition:

(5) Let us consider a heavy real number r. Then

(i) $\operatorname{sgn}(r) = \operatorname{sgn}(\lfloor r \rfloor)$, and

(ii) $\operatorname{sgn}(r) = \operatorname{sgn}(\lceil r \rceil)$.

Let us consider a real number a. Now we state the propositions:

(6) $\operatorname{sgn}(\operatorname{frac} a) = \operatorname{sgn}(\operatorname{frac}(-a))$.

(7) $\lfloor a \rfloor + \lfloor -a \rfloor + \operatorname{sgn}(\operatorname{frac} a) = 0$.

Let us consider a non integer real number a. Now we state the propositions:

(8) $\lfloor a \rfloor + \lfloor -a \rfloor = -1$. The theorem is a consequence of (7).

(9) $\operatorname{frac} a + \operatorname{frac}(-a) = 1$. The theorem is a consequence of (8).

Let us consider a real number r. Now we state the propositions:

(10) $\operatorname{frac}|r| = (\operatorname{sgn}(\operatorname{frac} r)) \cdot (\frac{1}{2} + (\operatorname{sgn}(r)) \cdot (\operatorname{frac} r - \frac{1}{2}))$. The theorem is a consequence of (9).

(11) $\lceil r \rceil = \lfloor r \rfloor + \operatorname{sgn}(\operatorname{frac} r)$.

Now we state the propositions:

(12) Let us consider integers a, b, c. Then $a \bmod b = b \cdot c + a \bmod b$.

(13) Let us consider non zero natural numbers a, b, and a natural number c. Then $(c \bmod a \cdot b) \bmod b = (c \bmod b) \bmod a \cdot b$.

Let us consider a real number a. Now we state the propositions:

(14) $\lfloor a \rfloor = -\lceil -a \rceil$. The theorem is a consequence of (8).

(15) $\lceil a \rceil = -\lfloor -a \rfloor$. The theorem is a consequence of (14).

Now we state the propositions:

(16) Let us consider a weightless, negative real number a. Then $\lfloor a \rfloor = -1$.

(17) Let us consider a weightless, positive real number a. Then $\lceil a \rceil = 1$.

(18) Let us consider a real number a. Then $\operatorname{frac} a = \operatorname{frac}(-a)$ if and only if $2 \cdot a$ is integer. The theorem is a consequence of (9).

(19) Let us consider integers a, b. Then $a \operatorname{div} -b = -a \operatorname{div} b$.

(20) Let us consider a real number a, and a non zero natural number b. Then $\operatorname{frac}(\frac{a}{b}) \cdot b < b$.

Let us consider integers a, b. Now we state the propositions:

(21) If $b \nmid a$, then $(a \bmod b) + (-a \bmod b) = b$. The theorem is a consequence of (9).

(22) $a \bmod b = -(-a \bmod -b)$.

(23) (i) $a \bmod -b = -(-a \bmod b)$, and

(ii) $-a \bmod b = -(a \bmod -b)$.

The theorem is a consequence of (22).

(24) If $b \nmid a$, then $(a \bmod b) - (a \bmod -b) = b$. The theorem is a consequence of (21) and (23).

Let us consider real numbers a, b. Now we state the propositions:

(25) $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor + \lfloor \operatorname{frac} a + \operatorname{frac} b \rfloor$.

(26) $\lfloor a \cdot b \rfloor = \lfloor a \rfloor \cdot \lfloor b \rfloor + \lfloor \lfloor a \rfloor \cdot \operatorname{frac} b + \lfloor b \rfloor \cdot \operatorname{frac} a + \operatorname{frac} a \cdot \operatorname{frac} b \rfloor$.

Now we state the proposition:

(27) Let us consider a real number a, and an integer b. Then $\lfloor a \cdot b \rfloor = \lfloor a \rfloor \cdot b + \lfloor \operatorname{frac} a \cdot b \rfloor$. The theorem is a consequence of (26).

Let us consider real numbers a, b. Now we state the propositions:

(28) If $a \leqslant b$, then $\lceil a \rceil \leqslant \lfloor b \rfloor + 1$.

(29) If $a \leqslant b$, then $\lfloor a \rfloor < \lfloor b \rfloor + 1$. The theorem is a consequence of (28).

(30) $\lfloor \frac{a}{b} \rfloor = \lfloor \frac{\lfloor a \rfloor}{b} \rfloor + \lfloor \mathrm{frac}(\frac{\lfloor a \rfloor}{b}) + \frac{\mathrm{frac}\, a}{b} \rfloor$.

Now we state the propositions:

(31) Let us consider a real number a, and a non zero natural number b. Then $\mathrm{frac}(\frac{\lfloor a \rfloor}{b}) + \frac{\mathrm{frac}\, a}{b} < 1$. The theorem is a consequence of (20).

(32) Let us consider a real number a, and a positive integer b. Then $\lfloor \frac{a}{b} \rfloor = \lfloor \frac{\lfloor a \rfloor}{b} \rfloor$. The theorem is a consequence of (31) and (30).

(33) Let us consider a real number a, and a negative integer b. Then $\lfloor \frac{a}{b} \rfloor = \lfloor \frac{\lceil a \rceil}{b} \rfloor$. The theorem is a consequence of (32) and (15).

(34) Let us consider a real number a, and a positive real number b. Then $-\lfloor \frac{a}{-b} \rfloor \geqslant \lfloor \frac{a}{b} \rfloor$. The theorem is a consequence of (15).

(35) Let us consider real numbers a, b. Suppose $\lfloor a \rfloor \leqslant b \leqslant \lceil a \rceil$. Then

(i) $\lfloor a \rfloor = \lfloor b \rfloor$, or

(ii) $\lceil a \rceil = \lceil b \rceil$.

The theorem is a consequence of (2).

(36) Let us consider a real number a, and a non integer real number b. Suppose $\lfloor a \rfloor \leqslant b \leqslant \lceil a \rceil$. Then

(i) $\lfloor a \rfloor = \lfloor b \rfloor$, and

(ii) $\lceil a \rceil = \lceil b \rceil$.

The theorem is a consequence of (35).

(37) Let us consider integers a, b, and a natural number c. Then $\lfloor \frac{a}{b \cdot c} \rfloor = \lfloor \frac{\lfloor \frac{a}{b} \rfloor}{c} \rfloor$. The theorem is a consequence of (32).

(38) Let us consider an integer a, a non zero integer b, and a negative integer c. Then $\lfloor \frac{a}{b \cdot c} \rfloor = \lfloor \frac{\lfloor -\frac{a}{b} \rfloor}{-c} \rfloor$. The theorem is a consequence of (37).

(39) Let us consider an integer a, and a non zero integer b. Suppose $\lfloor \frac{a}{b} \rfloor = -\lfloor -\frac{a}{b} \rfloor$. Then $b \mid a$. The theorem is a consequence of (7).

(40) Let us consider an integer a, and a negative integer b. Then $\lfloor \frac{a}{b} \rfloor = -(\lfloor \frac{a}{-b} \rfloor + \mathrm{sgn}(\mathrm{frac}(\frac{a}{b})))$. The theorem is a consequence of (7).

(41) Let us consider an integer a, a non zero integer b, and a non zero natural number c. Then $a \bmod b \cdot c = (a \bmod b) + b \cdot ((a \,\mathrm{div}\, b) \bmod c)$.

(42) Let us consider a natural number a, and non zero natural numbers b, c. Then $a \,\mathrm{div}\, c = ((a \,\mathrm{div}\, c) \bmod b) + b \cdot (a \,\mathrm{div}\, b \cdot c)$.

(43) Let us consider an integer a, a non zero natural number b, and natural numbers k, l. Then $a \bmod b^{k+l} = (a \bmod b^k) + ((a \,\mathrm{div}\, b^k) \bmod b^l) \cdot b^k$. The theorem is a consequence of (41).

(44) Let us consider natural numbers a, n, and a non zero natural number b. Then $a \bmod b^n = a - b^n \cdot (a \operatorname{div} b^n)$.

(45) Let us consider natural numbers a, b, n, k. Then $a \operatorname{div} b^{n+k} = (a \operatorname{div} b^n) \operatorname{div} b^k$.

Let us consider natural numbers a, n, k and a non zero natural number b. Now we state the propositions:

(46) $b^n \cdot ((a \operatorname{div} b^n) - (a \operatorname{div} b^{n+k}) \cdot b^k) = (a \bmod b^{n+k}) - (a \bmod b^n)$.

(47) $a \operatorname{div} b^n = (a \operatorname{div} b^{n+k}) \cdot b^k + ((a \operatorname{div} b^n) \bmod b^k)$. The theorem is a consequence of (43) and (46).

(48) $(a \operatorname{div} b^n) - (a \operatorname{div} b^{n+k}) \cdot b^k = (a \bmod b^{n+k}) - (a \bmod b^n) \operatorname{div} b^n$. The theorem is a consequence of (46).

Let us consider natural numbers a, n and a non zero natural number b. Now we state the propositions:

(49) $(a \operatorname{div} b^n) - (a \operatorname{div} b^{n+1}) \cdot b^1 = (a \bmod b^{n+1}) - (a \bmod b^n) \operatorname{div} b^n$.

(50) $(a \bmod b^{n+1}) \operatorname{div} b^n < b$.

(51) $(a \bmod b^{n+1}) - (a \bmod b^n) = b^n \cdot ((a \operatorname{div} b^n) - b \cdot (a \operatorname{div} b^{n+1}))$. The theorem is a consequence of (44).

Let us consider natural numbers a, n, k and a non zero natural number b. Now we state the propositions:

(52) $(a \bmod b^{n+k}) - (a \bmod b^n) = b^n \cdot ((a \operatorname{div} b^n) - b^k \cdot (a \operatorname{div} b^{n+k}))$. The theorem is a consequence of (44).

(53) $(a \bmod b^{n+k}) - (a \bmod b^n) = b^n \cdot ((a \operatorname{div} b^n) \bmod b^k)$. The theorem is a consequence of (52) and (45).

(54) $b^n \mid (a \bmod b^{n+k}) - (a \bmod b^n)$. The theorem is a consequence of (52).

(55) $(a \bmod b^{n+k}) - (a \bmod b^n) \bmod b^n = 0$. The theorem is a consequence of (54).

(56) $(a \bmod b^{n+k}) + b^{n+k} \cdot (a \operatorname{div} b^{n+k}) = (a \bmod b^n) + b^n \cdot (a \operatorname{div} b^n)$.

Now we state the proposition:

(57) Let us consider natural numbers a, b, n. Then $a \bmod b^n \leqslant a \bmod b^{n+1}$.

Let a, b, n be natural numbers. Let us observe that $(a \bmod b^{n+1}) - (a \bmod b^n)$ is natural.

Now we state the proposition:

(58) Let us consider a natural number a, and a natural number b. Then $a \operatorname{div} b^2 = (a \operatorname{div} b) \operatorname{div} b$.

Let us consider a natural number a and a non trivial natural number b. Now we state the propositions:

(59) $a \operatorname{div} b = (a \operatorname{div} b^0) - (a \bmod b^1) \operatorname{div} b$. The theorem is a consequence of (44).

(60) $a \operatorname{div} b^2 = (a \operatorname{div} b^0) - (a \bmod b^2) \operatorname{div} b^2$. The theorem is a consequence of (44).

Now we state the propositions:

(61) Let us consider natural numbers a, n, and a non zero natural number b. Then $(a \bmod b^{n+1}) - (a \bmod b^n) = ((a \bmod b^{n+1}) - (a \bmod b^n) \operatorname{div} b^n) \cdot b^n$.

(62) Let us consider a natural number a, and a non trivial natural number b. Then there exists a natural number n such that $a < b^n$.

(63) Let us consider non trivial natural numbers a, b. Then there exists a natural number n such that $b^n \leqslant a < b^{n+1}$.

2. Some Registrations for (X)FinSequences

Let n, k be natural numbers and f be an $(n + k)$-element, complex-valued finite 0-sequence. Let us note that $(f \restriction n) \frown \langle 0 \rangle$ is complex-valued.

Let n be a natural number and f be an $(n + 1)$-element, complex-valued finite 0-sequence. Let us observe that $(f \restriction n) \frown \langle f(n) \rangle$ reduces to f.

Let k be a complex number and f be a complex-valued finite 0-sequence. Let us note that $f \frown \langle k \rangle$ is complex-valued.

Let m, n be natural numbers, D be a non empty set, f be an m-element, D-valued finite 0-sequence, and g be an n-element, D-valued finite 0-sequence. One can verify that $(f \frown g) \restriction m$ reduces to f.

Now we state the propositions:

(64) Let us consider natural numbers n, i, and a complex number k. If $n \neq i$, then $((n \longmapsto 0) \frown \langle k \rangle)(i) = 0$.

(65) Let us consider a natural number n, and an $(n + 1)$-element, complex-valued finite 0-sequence f. Then $f = (f \restriction n) \frown \langle 0 \rangle + (n \longmapsto 0) \frown \langle f(n) \rangle$.
PROOF: Reconsider $g = (f \restriction n) \frown \langle 0 \rangle$ as an $(n+1)$-element, complex-valued finite 0-sequence. Reconsider $h = (n \longmapsto 0) \frown \langle f(n) \rangle$ as an $(n+1)$-element, complex-valued finite 0-sequence. For every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) = (g + h)(i)$ by [1, (44)], [11, (7)], [12, (51)]. $\square$

Let n be a natural number and f be an $(n + 1)$-element, complex-valued finite 0-sequence. Let us note that $(f \restriction n) \frown \langle 0 \rangle + (n \longmapsto 0) \frown \langle f(n) \rangle$ reduces to f.

3. Digits and Values

Let f be an empty yielding finite 0-sequence of $\mathbb{N}$ and b be a natural number. Note that $\operatorname{value}(f, b)$ is zero.

Let n be a natural number and b be a non trivial natural number. One can check that value(digits$(n, b), b)$ reduces to n.

Now we state the propositions:

(66) Let us consider natural numbers n, k, and a non trivial natural number b. Then $(\mathrm{digits}(n, b))(k) < b$.

(67) Let us consider natural numbers b, n, and n-element finite 0-sequences f, g of $\mathbb{N}$. Then $\mathrm{value}(f + g, b) = \mathrm{value}(f, b) + \mathrm{value}(g, b)$.

(68) Let us consider natural numbers b, k, n, and a k-element finite 0-sequence f of $\mathbb{N}$. Then $\mathrm{value}(f \frown \langle n \rangle, b) = \mathrm{value}(f, b) + n \cdot b^k$.

PROOF: Reconsider $g = f \frown \langle n \rangle$ as a $(k + 1)$-element finite 0-sequence of $\mathbb{N}$. Consider g_1 being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, g_1 = \mathrm{dom}\, g$ and for every natural number i such that $i \in \mathrm{dom}\, g_1$ holds $g_1(i) = g(i) \cdot b^i$ and $\mathrm{value}(g, b) = \sum g_1$. Consider f_1 being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, f_1 = \mathrm{dom}\, f$ and for every natural number i such that $i \in \mathrm{dom}\, f_1$ holds $f_1(i) = f(i) \cdot b^i$ and $\mathrm{value}(f, b) = \sum f_1$. $g_1 \restriction k = f_1$ by [12, (86)], [3, (49)]. □

(69) Let us consider natural numbers b, k, and a k-element finite 0-sequence f of $\mathbb{N}$. Then $\mathrm{value}(f \frown \langle 0 \rangle, b) = \mathrm{value}(f, b)$. The theorem is a consequence of (68).

(70) Let us consider natural numbers k, n, b. Then $\mathrm{value}(\langle n, k \rangle, b) = k \cdot b + n$.

(71) Let us consider natural numbers k, n, and a natural number b. Then $\mathrm{value}((k \longmapsto 0) \frown \langle n \rangle, b) = n \cdot b^k$.

PROOF: Consider x being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, x = \mathrm{dom}((k \longmapsto 0) \frown \langle n \rangle)$ and for every natural number i such that $i \in \mathrm{dom}\, x$ holds $x(i) = ((k \longmapsto 0) \frown \langle n \rangle)(i) \cdot b^i$ and $\mathrm{value}((k \longmapsto 0) \frown \langle n \rangle, b) = \sum x$. For every natural number i such that $i \in \mathrm{dom}(x \restriction k)$ holds $(x \restriction k)(i) = (k \longmapsto 0)(i)$ by [12, (86), (53)], (133). □

(72) Let us consider natural numbers k, m, n, and a natural number b. Then $\mathrm{value}(\langle n, m, k \rangle, b) = k \cdot b^2 + m \cdot b + n$. The theorem is a consequence of (68) and (70).

(73) Let us consider a non zero natural number b, and a finite 0-sequence f of $\mathbb{N}$. If $\mathrm{value}(f, b) = 0$, then f is empty yielding.

PROOF: Consider x being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, x = \mathrm{dom}\, f$ and for every natural number i such that $i \in \mathrm{dom}\, x$ holds $x(i) = f(i) \cdot b^i$ and $\mathrm{value}(f, b) = \sum x$. For every natural number k such that $k \in \mathrm{dom}\, x$ holds $f(k) = 0$ by [1, (3)], [5, (61)]. □

Let us consider a natural number n and a non trivial natural number b. Now we state the propositions:

(74) If $\mathrm{digits}(n, b) = \langle n \rangle$, then $n < b$.

(75) If $n \geqslant b$, then $\mathrm{len}(\mathrm{digits}(n, b)) > 1$. The theorem is a consequence of (74).

(76) $(\mathrm{digits}(n, b))(0) = n \bmod b$. The theorem is a consequence of (66).

(77) $n \operatorname{div} b = \mathrm{value}(\mathrm{mid}(\mathrm{digits}(n, b), 2, \mathrm{len}(\mathrm{digits}(n, b))), b)$. The theorem is a consequence of (76).

Now we state the propositions:

(78) Let us consider a natural number a, a non trivial natural number b, and a non zero natural number c. Then $\mathrm{value}(\mathrm{digits}(a, b), b) \bmod c = \mathrm{value}(\mathrm{digits}(a \bmod c, b), b)$.

(79) Let us consider a natural number a, a non trivial natural number b, and a non zero natural number n. Then $\mathrm{value}(\mathrm{digits}(a, b^n), b^n) \bmod b = \mathrm{value}(\mathrm{digits}(a \bmod b, b^n), b^n)$.

4. On Integer Numbers

Let a be a non negative real number. Note that $\sqrt{a^{\mathbf{2}}}$ reduces to a and $\sqrt{a}^{\mathbf{2}}$ reduces to a and $\sqrt{a}^2$ reduces to a and $\sqrt{a} \cdot \sqrt{a}$ reduces to a.

Let a be a natural number and b be an integer. Let us note that $\frac{a}{\gcd(a,b)}$ is natural.

One can check that $\gcd(a \cdot b, a)$ reduces to a.

Now we state the propositions:

(80) Let us consider an integer a, and a non zero integer b. Then $\frac{a}{\gcd(b,a)}$ and $\frac{b}{\gcd(a,b)}$ are relatively prime.

(81) Let us consider natural numbers a, n, and a real number b. If $(a+1)^n > b^n > a^n$, then b is not integer.

(82) Let us consider a non integer real number a. Then $\lfloor a \rfloor + \lfloor -a \rfloor = -1$.

Let a be a non integer real number. Let us observe that $\lfloor a \rfloor + \lfloor -a \rfloor$ is weightless and negative.

Now we state the proposition:

(83) Let us consider a real number a. Then $\lfloor a \rfloor + \lfloor -a \rfloor$ is weightless.

5. On Rational Numbers

One can check that there exists a positive rational number which is light and non integer.

Now we state the proposition:

(84) Let us consider non zero natural numbers a, b. Then

(i) $\operatorname{den} \frac{a}{b} \leqslant b$, and

(ii) $\operatorname{num} \frac{a}{b} \leqslant a$.

Let us consider a rational number a. Now we state the propositions:

(85) $\operatorname{den} \operatorname{den} a = 1$.

(86) $\operatorname{den} \operatorname{num} a = 1$.

Now we state the proposition:

(87) Let us consider a non trivial natural number a, and non zero natural numbers b, c. Suppose $a \mid b$ and $a \mid c$. Then

(i) $\operatorname{den} \frac{b}{c} < c$, and

(ii) $\operatorname{num} \frac{b}{c} < b$.

The theorem is a consequence of (84).

Let a be a rational number. Note that $\gcd(\operatorname{den} a, \operatorname{num} a)$ is trivial and non zero.

Now we state the propositions:

(88) Let us consider an integer a, and a non zero integer b. Then

(i) $\operatorname{den} \frac{a}{b} \mid b$, and

(ii) $\operatorname{num} \frac{a}{b} \mid a$.

(89) Let us consider an integer a, and a non zero natural number b. If a and b are relatively prime, then $\operatorname{den} \frac{a}{b} = b$.

(90) Let us consider a non zero, light rational number a. Then $\operatorname{den} a \nmid \operatorname{num} a$.

Let us consider a rational number a. Now we state the propositions:

(91) $\lfloor a \rfloor = \operatorname{num} a \operatorname{div} \operatorname{den} a$.

(92) $\operatorname{frac} a = \frac{\operatorname{num} a \bmod \operatorname{den} a}{\operatorname{den} a}$. The theorem is a consequence of (91).

(93) $\operatorname{frac} a \cdot (\operatorname{den} a) = \operatorname{num} a \bmod \operatorname{den} a$. The theorem is a consequence of (92).

Let a be a rational number. Observe that $\operatorname{frac} a$ is rational and $\operatorname{frac} a \cdot (\operatorname{den} a)$ is natural.

Let us consider a rational number a. Now we state the propositions:

(94) $\operatorname{num} \operatorname{frac} a$ and $\operatorname{den} \operatorname{frac} a$ are relatively prime.

(95) $\operatorname{num} \operatorname{frac} a \bmod \operatorname{den} \operatorname{frac} a = \operatorname{num} \operatorname{frac} a$.

(96) $\operatorname{num} a \bmod \operatorname{den} a$ and $\operatorname{den} a$ are relatively prime.

(97) $\operatorname{den} \operatorname{frac} a = \operatorname{den} \operatorname{frac}(-a)$. The theorem is a consequence of (92) and (96).

(98) $\operatorname{den} a = \operatorname{den} \operatorname{frac} a$. The theorem is a consequence of (96), (92), and (97).

Now we state the propositions:

(99) Let us consider a rational number a, and an integer b. Then $\operatorname{den}(a+b) = \operatorname{den} a$. The theorem is a consequence of (98).

(100) Let us consider a negative, non integer rational number a. Then $\operatorname{frac} a = 1 - \frac{\operatorname{num}(-a) \bmod \operatorname{den} a}{\operatorname{den} a}$. The theorem is a consequence of (92).

(101) Let us consider a positive rational number a. Then

(i) $\operatorname{num} a = \operatorname{den} a^{-1}$, and

(ii) $\operatorname{den} a = \operatorname{num} a^{-1}$.

Let a be a positive natural number. One can verify that $\operatorname{num} a^{-1}$ is weightless and positive.

One can check that $\operatorname{den} a^{-1}$ reduces to a.

Now we state the proposition:

(102) Let us consider a positive rational number a. Then $(\operatorname{num} a) \cdot (\operatorname{num} a^{-1}) = (\operatorname{den} a) \cdot (\operatorname{den} a^{-1})$. The theorem is a consequence of (101).

Let us consider rational numbers a, b. Now we state the propositions:

(103) $\frac{\operatorname{num} a \cdot b}{\operatorname{den} a \cdot b} = \frac{(\operatorname{num} a) \cdot (\operatorname{num} b)}{(\operatorname{den} a) \cdot (\operatorname{den} b)}$.

(104) $(\operatorname{num} a \cdot b) \cdot (\operatorname{den} a) \cdot (\operatorname{den} b) = (\operatorname{den} a \cdot b) \cdot (\operatorname{num} a) \cdot (\operatorname{num} b)$. The theorem is a consequence of (103).

(105) $\operatorname{num} a \cdot b \mid (\operatorname{num} a) \cdot (\operatorname{num} b)$. The theorem is a consequence of (104).

(106) $\operatorname{den} a \cdot b \mid (\operatorname{den} a) \cdot (\operatorname{den} b)$. The theorem is a consequence of (104).

Now we state the proposition:

(107) Let us consider non zero rational numbers a, b. Then $\frac{(\operatorname{den} a) \cdot (\operatorname{den} b)}{\operatorname{den} a \cdot b} = \frac{(\operatorname{num} a) \cdot (\operatorname{num} b)}{\operatorname{num} a \cdot b}$.

Let us consider rational numbers a, b. Now we state the propositions:

(108) If $a \cdot b$ is integer, then $\operatorname{den} b \mid \operatorname{num} a$.

(109) $a \cdot b$ is integer if and only if $\operatorname{den} b \mid \operatorname{num} a$ and $\operatorname{den} a \mid \operatorname{num} b$.

PROOF: If $\operatorname{den} b \mid \operatorname{num} a$ and $\operatorname{den} a \mid \operatorname{num} b$, then $a \cdot b$ is integer by [14, (1)]. $\square$

Now we state the propositions:

(110) Let us consider non negative rational numbers a, b. Then $(\operatorname{num} a) \cdot (\operatorname{num} b) \geqslant \operatorname{num} a \cdot b$. The theorem is a consequence of (105).

(111) Let us consider rational numbers a, b. Then $(\operatorname{den} a) \cdot (\operatorname{den} b) \geqslant \operatorname{den} a \cdot b$. The theorem is a consequence of (106).

(112) Let us consider a rational number a, and a non zero natural number k. If $a \cdot k$ is integer, then $k \geqslant \operatorname{den} a$.

(113) Let us consider non zero rational numbers a, b. Then $(\operatorname{num}(a+b)) \cdot (\operatorname{den} a) \cdot (\operatorname{den} b) = ((\operatorname{num} b) \cdot (\operatorname{den} a) + (\operatorname{num} a) \cdot (\operatorname{den} b)) \cdot (\operatorname{den}(a+b))$.

6. On Irrational Numbers

Now we state the propositions:

(114) Let us consider a non zero natural number n, and a non integer rational number a. Then a^n is not integer. The theorem is a consequence of (80).

(115) Let us consider natural numbers a, b, n. If $(a+1)^n > b > a^n$, then $\sqrt[n]{b}$ is irrational. The theorem is a consequence of (81) and (114).

(116) Let us consider a natural number a, and a non zero natural number n. Then $\sqrt[n]{a}$ is irrational if and only if $\sqrt[n]{a}$ is not integer.

Proof: If $\sqrt[n]{a}$ is not integer, then $\sqrt[n]{a}$ is irrational by [1, (14)], [8, (7)], [11, (29), (26)]. □

Let a be a non square natural number. Let us note that $\sqrt[2]{a}$ is irrational.

Let n be a non zero, even natural number. Let us observe that $\sqrt[n]{a}$ is irrational.

Now we state the proposition:

(117) Let us consider a prime natural number a, and non zero natural numbers n, m. Then $\sqrt[n+m]{a^m}$ is irrational.

Let us consider a non zero natural number n and natural numbers a, b. Now we state the propositions:

(118) If a and b are relatively prime, then if $\sqrt[n]{a \cdot b}$ is a natural number, then $\sqrt[n]{a}$ is a natural number.

(119) If a and b are relatively prime, then if $\sqrt[n]{a \cdot b}$ is rational, then $\sqrt[n]{a}$ is rational. The theorem is a consequence of (118).

Now we state the propositions:

(120) Let us consider a prime natural number a, and non zero natural numbers b, n, m. If $a \nmid b$, then $\sqrt[n+m]{a^m \cdot b}$ is irrational. The theorem is a consequence of (117) and (119).

(121) Let us consider a non trivial natural number n, a prime natural number a, and a natural number b. If $a \mid b$ and $a^n \nmid b$, then $\sqrt[n]{b}$ is irrational. The theorem is a consequence of (120).

Let a be a non trivial, square-free natural number and n be a non trivial natural number. One can check that $\sqrt[n]{a}$ is irrational.

Let a be a prime natural number and b be a non zero, rational number. Observe that $\sqrt{a} \cdot (\sqrt{a} + b)$ is irrational.

Now we state the proposition:

(122) Let us consider a prime natural number a, and rational numbers b, c. Then $(\sqrt{a} + b) \cdot (\sqrt{a} + c)$ is rational if and only if $b = -c$.

Let a be a prime natural number and b be a rational number. One can verify that $(\sqrt{a}+b)\cdot(\sqrt{a}-b)$ is rational.

Let b be a non zero, rational number. Let us observe that $(\sqrt{a}+b)\cdot(\sqrt{a}+b)$ is irrational.

Let b, c be positive, rational numbers. Let us observe that $(\sqrt{a}+b)\cdot(\sqrt{a}+c)$ is irrational and $(\sqrt{a}-b)\cdot(\sqrt{a}-c)$ is irrational.

Now we state the proposition:

(123) Let us consider a non zero natural number n, and a non negative real number a. If $a > 1$, then $\sqrt[n]{a} > 1$.

Let n be a non zero natural number and a be a trivial natural number. Let us observe that $\sqrt[n]{a}$ is natural.

7. Square-free Numbers and Radicals

Let a be a non trivial, square-free natural number and n, m be non zero natural numbers. Observe that $\sqrt[n+m]{a^m}$ is irrational.

Let b be a non zero, rational number. Let us note that $\sqrt[n+m]{a^n}\cdot(\sqrt[n+m]{a^m}+b)$ is irrational and $\sqrt{a}$ is irrational.

Let b be a non zero, rational number. Observe that $\sqrt{a}\cdot(\sqrt{a}+b)$ is irrational.

Now we state the proposition:

(124) Let us consider natural numbers a, b. Then $a\cdot b = (\gcd(a,b))^2\cdot\frac{\mathrm{lcm}(a,b)}{\gcd(a,b)}$.

Let a be a non zero natural number. Let us observe that $\frac{a}{\mathrm{Rad}(a)}$ is natural.

Let a be an integer and b be a square-free natural number. Note that $\gcd(a,b)$ is square-free.

Now we state the proposition:

(125) Let us consider a non zero, square-containing natural number a. Then $a > \mathrm{Rad}(a)$.

Let a be a non zero, square-containing natural number. One can check that $\frac{a}{\mathrm{Rad}(a)}$ is non trivial.

Now we state the proposition:

(126) Let us consider an integer a. Suppose $a \neq 1$ and $a \neq -1$. Then there exists a prime natural number p such that $p \mid a$.

Let a be a non trivial, a square natural number and b be a natural number. One can check that $a\cdot b$ is square-containing.

Let a be a natural number and b be a square-containing natural number. Let us observe that $\mathrm{lcm}(a,b)$ is square-containing.

Now we state the propositions:

(127) Let us consider natural numbers a, b. If a and b are not relatively prime, then $a\cdot b$ is square-containing. The theorem is a consequence of (124).

(128) Let us consider a square-free natural number a, and a natural number b. Then $\gcd(a, b)$ and $\frac{a}{\gcd(a,b)}$ are relatively prime. The theorem is a consequence of (127).

(129) Let us consider a square-free natural number b, and a natural number a. If $b \nmid a$, then there exists a prime natural number p such that $p \nmid a$ and $p \mid b$. The theorem is a consequence of (128).

8. IRRATIONAL ROOTS

Now we state the propositions:

(130) Let us consider a square-free natural number a, and non zero natural numbers b, n, m. If $a \nmid b$, then $\sqrt[n+m]{a^m \cdot b}$ is irrational. The theorem is a consequence of (129), (128), and (119).

(131) Let us consider a square-free natural number a, a non zero natural number b, and a non trivial natural number n. If $b < a^n$ and $a \mid b$, then $\sqrt[n]{b}$ is irrational. The theorem is a consequence of (130).

Let a be a non zero natural number and n be a non trivial natural number. Observe that $\sqrt[n]{a^n + 1}$ is irrational.

Let a, n be non trivial natural numbers. Let us observe that $\sqrt[n]{a^n - 1}$ is irrational.

Now we state the propositions:

(132) Let us consider a natural number b, and a non zero natural number n. Suppose $\sqrt[n]{b}$ is irrational. Then there exists a natural number a such that $a^n < b < (a + 1)^n$.

(133) Let us consider natural numbers a, b, and a non zero natural number n. Suppose $\sqrt[n]{b}$ is irrational and $\sqrt[n]{b} > a$. Then there exists a natural number c such that $a^n < b < (a + 2 \cdot c)^n$. The theorem is a consequence of (132).

(134) Let us consider a natural number a, and a non zero natural number n. Suppose $\sqrt[n]{a}$ is irrational. Then there exists a natural number b such that $1 < a < (2 \cdot b + 1)^n$. The theorem is a consequence of (123) and (133).

(135) Let us consider a non zero natural number a, a non trivial natural number n, and a natural number m. Then $\sqrt[n]{a^{n \cdot m} + 1}$ is irrational.

(136) Let us consider non trivial natural numbers a, n, and a non zero natural number m. Then $\sqrt[n]{a^{n \cdot m} - 1}$ is irrational.

(137) Let us consider a non zero natural number b, and a non trivial natural number n. If $\sqrt[2]{b}$ is natural, then $\sqrt[2]{b + 2}$ is irrational.

(138) Let us consider a non zero natural number b. If $\sqrt[2]{b + 2}$ is natural, then $\sqrt[2]{b}$ is irrational. The theorem is a consequence of (114) and (137).

(139) Let us consider a prime natural number a, a natural number m, and a non trivial natural number n. Then $\sqrt[n]{a^m}$ is rational if and only if $n \mid m$.
PROOF: If $\sqrt[n]{a^m}$ is rational, then $n \mid m$ by [11, (62)], [1, (10)], [4, (8)], [8, (11)]. □

(140) Let us consider a non trivial, square-free, odd natural number a, and a non zero natural number b. If a and b are relatively prime, then $\sqrt[2]{a \cdot (2 \cdot b + a)}$ is irrational. The theorem is a consequence of (130).

(141) Let us consider a square-free, odd natural number a, and a non zero natural number b. Suppose a and b are relatively prime and $\sqrt[2]{a \cdot (2 \cdot b + a)}$ is rational. Then a is trivial.

(142) Let us consider a non trivial, square-free natural number a, and a non trivial natural number n. Then $\sqrt[n]{a}$ is irrational.

(143) Let us consider a prime natural number a, and a non trivial natural number n. Then $\sqrt[n]{a}$ is irrational.

(144) Let us consider a non zero natural number b. Then $\sqrt[2]{b^2 + 2}$ is irrational. The theorem is a consequence of (137).

Let us consider non zero natural numbers a, b. Now we state the propositions:

(145) If $b \leqslant 2 \cdot a$, then $\sqrt[2]{a^2 + b}$ is irrational. The theorem is a consequence of (115).

(146) If $\sqrt[2]{a^2 + b^2}$ is natural, then $b^2 > 2 \cdot a$.

Now we state the propositions:

(147) Let us consider natural numbers a, b. If $a > b$, then $\sqrt[2]{a^2 + 2 \cdot b + 1}$ is irrational. The theorem is a consequence of (115).

(148) Let us consider non zero natural numbers a, b. If $b \leqslant 2 \cdot a$, then $\sqrt[2]{(a+1)^2 - b}$ is irrational. The theorem is a consequence of (115).

(149) Let us consider a prime natural number a, a non zero natural number b, and a non trivial natural number n. If $b < a^n$ and $a \mid b$, then $\sqrt[n]{b}$ is irrational.

(150) Let us consider non zero natural numbers a, b, and a non trivial natural number n. Suppose $b < (\mathrm{Rad}(a))^n$ and $\mathrm{Rad}(a) \mid b$. Then $\sqrt[n]{b}$ is irrational.

(151) Let us consider a prime natural number a, a non trivial natural number n, and natural numbers b, m. If $a \nmid b$ and $n \nmid m$, then $\sqrt[n]{a^m \cdot b}$ is irrational. The theorem is a consequence of (139) and (119).

(152) Let us consider a square-free natural number a, and a non trivial natural number n. Then $\sqrt[n]{a}$ is a rational number if and only if $a = 1$.

(153) Let us consider a natural number a, and a non zero natural number n. Suppose $\sqrt[n]{a}$ is rational. Then

(i) $\sqrt[n]{\mathrm{Parity}(a)}$ is rational, and

(ii) $\sqrt[n]{\mathrm{Oddity}(a)}$ is rational.

The theorem is a consequence of (119).

(154) Let us consider natural numbers a, b. Then $(a+\sqrt{2}\cdot(1+\sqrt{2})\cdot b)\cdot(a+\sqrt{2}\cdot(\sqrt{2}-1)\cdot b)$ is rational.

(155) $\sqrt[2]{123^4-1}$ is irrational.

(156) $\sqrt[3]{678^9+1}$ is irrational. The theorem is a consequence of (135).

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