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Suszko's Non-Fregean Logics. Part I[1]

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Summary. The basic properties of non-Fregean logics in general and of Sentential Calculus with Identity in particular, as introduced by Roman Suszko in [4] and [5].

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1. Preliminaries

From now on k, m, n denote elements of $\mathbb{N}$, i, j denote natural numbers, a, b, c denote objects, y, z denote sets, and p, q, r, s denote finite sequences.

The functor VARS yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all $\langle 0, k\rangle$ where k is an element of $\mathbb{N}$.

Observe that VARS is non empty and antichain-like.

A variable is an element of VARS. The functors: $'\mathrm{not}'$, $'\wedge'$, and $'\vee'$ yielding finite sequences are defined by terms

(Def. 2) $\langle 11\rangle$,

(Def. 3) $\langle 21\rangle$,

(Def. 4) $\langle 22\rangle$,

respectively. The functors: $'\rightarrow'$, $'\leftrightarrow'$, and $'\equiv'$ yielding finite sequences are defined by terms

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(Def. 5) $\langle 23 \rangle$,

(Def. 6) $\langle 24 \rangle$,

(Def. 7) $\langle 25 \rangle$,

respectively. The functors: SCI-unops and SCI-binops yielding non empty, finite sequence-membered sets are defined by terms

(Def. 8) $\{'\mathrm{not}'\}$,

(Def. 9) $\{'\wedge', '\vee', '\rightarrow', '\leftrightarrow', '\equiv'\}$,

respectively. Now we state the proposition:

(1) (i) $a \in$ SCI-unops iff $a = '\mathrm{not}'$, and

(ii) $a \in$ SCI-binops iff $a = '\wedge'$ or $a = '\vee'$ or $a = '\rightarrow'$ or $a = '\leftrightarrow'$ or $a = '\equiv'$.

Let F, G be non empty, finite sequence-membered sets. One can verify that $F \cup G$ is non empty and finite sequence-membered.

The functor SCI-ops yielding a non empty, finite sequence-membered set is defined by the term

(Def. 10) SCI-unops $\cup$ SCI-binops.

Now we state the proposition:

(2) (i) if $p = '\mathrm{not}'$, then $p(1) = 11$, and

(ii) if $p = '\wedge'$, then $p(1) = 21$, and

(iii) if $p = '\vee'$, then $p(1) = 22$, and

(iv) if $p = '\rightarrow'$, then $p(1) = 23$, and

(v) if $p = '\leftrightarrow'$, then $p(1) = 24$, and

(vi) if $p = '\equiv'$, then $p(1) = 25$.

One can verify that SCI-ops is non empty and antichain-like.

The functor SCI-symbols yielding a non empty, finite sequence-membered set is defined by the term

(Def. 11) VARS $\cup$ SCI-ops.

The functors: VARS, SCI-ops, SCI-unops, and SCI-binops yield non empty subsets of SCI-symbols. The functors: $'\mathrm{not}'$, $'\wedge'$, $'\vee'$, $'\rightarrow'$, and $'\leftrightarrow'$ yield elements of SCI-symbols. Observe that SCI-symbols is non trivial and antichain-like.

Note that the functor SCI-symbols yields a non trivial Polish language. The functor SCI-op-arity yielding a function from SCI-ops into $\mathbb{N}$ is defined by the term

(Def. 12) (SCI-binops $\longmapsto 2) +\cdot$ (SCI-unops $\longmapsto 1)$.

The functor SCI-arity yielding a Polish arity-function of SCI-symbols is defined by the term

(Def. 13) SCI-op-arity $+\cdot$(VARS $\longmapsto$ 0).

Now we state the propositions:

(3) If $a \in$ VARS, then (SCI-arity)(a) = 0.

(4) (i) (SCI-arity)($'$not$'$) = 1, and

(ii) for every a such that $a \in$ SCI-binops holds (SCI-arity)(a) = 2.

(5) The Polish atoms(SCI-symbols , SCI-arity) = VARS. The theorem is a consequence of (4) and (3).

The functor SCI-formula-set yielding a full Polish language of SCI-symbols is defined by the term

(Def. 14) Polish-WFF-set(SCI-symbols, SCI-arity).

An SCI-formula is a Polish WFF of SCI-symbols and SCI-arity. Let us observe that there exists a subset of SCI-formula-set which is non empty.

Let us consider n. The functor x_n yielding an SCI-formula is defined by the term

(Def. 15) $\langle 0, n \rangle$.

In the sequel X denotes an extension of SCI-arity, L denotes a Polish-ext-set of X, and t, u, v, w denote formulae of L.

Let us consider X. Now we state the propositions:

(6) SCI-symbols $\subseteq \mathrm{dom}\, X$.

(7) (i) $'$not$' \in \mathrm{dom}\, X$, and

(ii) $X('\mathrm{not}') = 1$, and

(iii) for every a such that $a \in$ SCI-binops holds $a \in \mathrm{dom}\, X$ and $X(a) = 2$.
The theorem is a consequence of (6) and (4).

Let us consider X, L, and n. The functor x.(n, L) yielding a formula of L is defined by the term

(Def. 16) x_n.

Now we state the proposition:

(8) If $m \neq n$, then $\mathrm{x}_m \neq \mathrm{x}_n$.

Let us consider p. The functor $\neg p$ yielding a finite sequence is defined by the term

(Def. 17) $'\mathrm{not}' \frown p$.

Let us consider q. The functors: $p \wedge q$, $p \vee q$, $p \rightarrow q$, $p \leftrightarrow q$, and $p \equiv q$ yielding finite sequences are defined by terms

(Def. 18) $'\wedge' \frown (p \frown q)$,

(Def. 19) $'\vee' \frown (p \frown q)$,

(Def. 20) $'\rightarrow' \frown (p \frown q)$, (Def. 21) $' \leftrightarrow ' \frown (p \frown q)$,

(Def. 22) $' \equiv ' \frown (p \frown q)$,

respectively. Let us consider X, L, and t. One can check that the functor $\neg t$ is defined by the term

(Def. 23) $(\text{Polish-unOp}(X, L, '\text{not}'))(t)$.

Let us consider u. The functors: $t \wedge u$, $t \vee u$, $t \rightarrow u$, $t \leftrightarrow u$, and $t \equiv u$ are defined by terms

(Def. 24) $(\text{Polish-binOp}(X, L, '\wedge'))(t, u)$,

(Def. 25) $(\text{Polish-binOp}(X, L, '\vee'))(t, u)$,

(Def. 26) $(\text{Polish-binOp}(X, L, '\rightarrow'))(t, u)$,

(Def. 27) $(\text{Polish-binOp}(X, L, '\leftrightarrow'))(t, u)$,

(Def. 28) $(\text{Polish-binOp}(X, L, '\equiv'))(t, u)$,

respectively. Note that the functor $\neg t$ yields a formula of L. Let us consider u. The functors: $t \wedge u$, $t \vee u$, $t \rightarrow u$, $t \leftrightarrow u$, and $t \equiv u$ yield formulae of L. The functor $t \Rightarrow u$ yielding a formula of L is defined by the term

(Def. 29) $t \equiv t \wedge u$.

Let u be an SCI-formula. The functors: $t \rightarrow u$ and $t \equiv u$ yield formulae of L. The functors: $u \rightarrow t$ and $u \equiv t$ yield formulae of L. We say that t is atomic if and only if

(Def. 30) $t \in$ the Polish atoms(SCI-symbols , SCI-arity).

We say that t is negative if and only if

(Def. 31) $\text{PolishExtHead}(t) = '\text{not}'$.

We say that t is conjunctive if and only if

(Def. 32) $\text{PolishExtHead}(t) = '\wedge'$.

We say that t is disjunctive if and only if

(Def. 33) $\text{PolishExtHead}(t) = '\vee'$.

We say that t is conditional if and only if

(Def. 34) $\text{PolishExtHead}(t) = '\rightarrow'$.

We say that t is biconditional if and only if

(Def. 35) $\text{PolishExtHead}(t) = '\leftrightarrow'$.

We say that t is an equality if and only if

(Def. 36) $\text{PolishExtHead}(t) = '\equiv'$.

Let us consider t. Now we state the propositions:

(9) t is atomic if and only if $t \in \text{VARS}$.

(10) t is negative if and only if there exists u such that $t = \neg u$.

PROOF: $'\mathrm{not}' \in \mathrm{dom}\, X$ and $X('\mathrm{not}') = 1$. If t is negative, then there exists u such that $t = \neg u$ by [1, (9)]. □

(11) t is conjunctive if and only if there exists u and there exists v such that $t = u \wedge v$.

PROOF: $'\wedge' \in \mathrm{dom}\, X$ and $X('\wedge') = 2$. If t is conjunctive, then there exists u and there exists v such that $t = u \wedge v$ by [1, (11)]. □

2. SCI Axioms

Let us consider X and L. The functors: SCI-prop-axioms L and SCI-id-axioms L yielding non empty subsets of L are defined by conditions

(Def. 37) for every a, $a \in$ SCI-prop-axioms L iff there exists t and there exists u and there exists v such that $a = t \to (u \to t)$ or $a = (t \to (u \to v)) \to ((t \to u) \to (t \to v))$ or $a = (\neg t \to \neg u) \to (u \to t)$ or $a = t \wedge u \to \neg(t \to \neg u)$ or $a = \neg(t \to \neg u) \to t \wedge u$ or $a = (t \vee u) \to (\neg t \to u)$ or $a = (\neg t \to u) \to (t \vee u)$ or $a = (t \leftrightarrow u) \to (t \to u) \wedge (u \to t)$ or $a = (t \to u) \wedge (u \to t) \to (t \leftrightarrow u)$,

(Def. 38) for every a, $a \in$ SCI-id-axioms L iff there exists t and there exists u and there exists v and there exists w such that $a = t \equiv t$ or $a = (t \equiv u) \to (\neg t \equiv \neg u)$ or $a = (t \equiv u) \wedge (v \equiv w) \to (t \wedge v \equiv u \wedge w)$ or $a = (t \equiv u) \wedge (v \equiv w) \to ((t \vee v) \equiv (u \vee w))$ or $a = (t \equiv u) \wedge (v \equiv w) \to ((t \to v) \equiv (u \to w))$ or $a = (t \equiv u) \wedge (v \equiv w) \to ((t \leftrightarrow v) \equiv (u \leftrightarrow w))$ or $a = (t \equiv u) \wedge (v \equiv w) \to ((t \equiv v) \equiv (u \equiv w))$ or $a = (t \equiv u) \to (t \to u)$,

respectively. Let B be a subset of L. Observe that there exists a non empty subset of L which is B-extending.

The functor SCI-axioms L yielding a (SCI-prop-axioms L)-extending subset of L is defined by the term

(Def. 39) SCI-prop-axioms $L \cup$ SCI-id-axioms L.

From now on R, R_1, R_2 denote rules of L.

Let us consider X and L. The functor SCI-MP L yielding a rule of L is defined by the term

(Def. 40) the set of all $\langle \{t, t \to u\}, u \rangle$ where t, u are formulae of L.

The functor SCI-rules L yielding a rule of L is defined by the term

(Def. 41) SCI-MP L.

A formula-sequence of L is a finite sequence of elements of L.

A formula-finset of L is a finite subset of L. In the sequel A, A_1, A_2 denote non empty subsets of L, B, B_1, B_2 denote subsets of L, P, P_1, P_2 denote formula-sequences of L, and S, S_1, S_2 denote formula-finsets of L.

Let us consider X, L, and t. One can verify that the functor $\{t\}$ yields a formula-finset of L.

3. Provability

Let us consider X, L, B, and a. We say that a is B-provable if and only if

(Def. 42) a is (B, (SCI-rules L))-provable.

Observe that every object which is B-axiomatic is also B-provable.

Now we state the proposition:

(12) If t is B-provable and $t \to u$ is B-provable, then u is B-provable.

Let us consider X, L, and a. We say that a is L-prop-axiomatic if and only if

(Def. 43) a is (SCI-prop-axioms L)-axiomatic.

We say that a is L-id-axiomatic if and only if

(Def. 44) a is (SCI-id-axioms L)-axiomatic.

We say that a is L-SCI-axiomatic if and only if

(Def. 45) a is (SCI-axioms L)-axiomatic.

We say that a is L-SCI-provable if and only if

(Def. 46) a is (SCI-axioms L)-provable.

One can verify that every element of SCI-prop-axioms L is L-prop-axiomatic and every element of SCI-id-axioms L is L-id-axiomatic and every element of SCI-axioms L is L-SCI-axiomatic and every object which is L-SCI-axiomatic is also L-SCI-provable and every object which is L-prop-axiomatic is also L-SCI-axiomatic and every object which is L-id-axiomatic is also L-SCI-axiomatic.

Let us consider t. Observe that $t \equiv t$ is L-id-axiomatic.

Let us consider u. One can verify that $t \to (u \to t)$ is L-prop-axiomatic and $(\neg t \to \neg u) \to (u \to t)$ is L-prop-axiomatic and $t \wedge u \to \neg(t \to \neg u)$ is L-prop-axiomatic and $\neg(t \to \neg u) \to t \wedge u$ is L-prop-axiomatic and $(t \vee u) \to (\neg t \to u)$ is L-prop-axiomatic and $(\neg t \to u) \to (t \vee u)$ is L-prop-axiomatic and $(t \leftrightarrow u) \to (t \to u) \wedge (u \to t)$ is L-prop-axiomatic and $(t \to u) \wedge (u \to t) \to (t \leftrightarrow u)$ is L-prop-axiomatic and $(t \equiv u) \to (\neg t \equiv \neg u)$ is L-id-axiomatic and $(t \equiv u) \to (t \to u)$ is L-id-axiomatic.

Let us consider v. Observe that $(t \to (u \to v)) \to ((t \to u) \to (t \to v))$ is L-prop-axiomatic.

Let us consider w. Let O be an element of SCI-binops. Note that $(t \equiv u) \wedge (v \equiv w) \to ((\text{Polish-binOp}(X, L, O))(t, v) \equiv (\text{Polish-binOp}(X, L, O))(u, w))$ is L-id-axiomatic and there exists a formula of L which is L-prop-axiomatic and there exists a formula of L which is L-id-axiomatic.

In the sequel C denotes a (SCI-prop-axioms L)-extending subset of L.

Let us consider X and L. Let us note that every formula of L which is L-prop-axiomatic is also (SCI-prop-axioms L)-provable.

Let us consider C. Let us note that every formula of L which is non C-provable is also non (SCI-prop-axioms L)-provable and there exists a formula of L which is C-provable.

Let us consider t. Let u be a C-provable formula of L. Observe that $t \to u$ is C-provable.

Now we state the propositions:

(13) If $t \to u$ is C-provable and $u \to v$ is C-provable, then $t \to v$ is C-provable. The theorem is a consequence of (12).

(14) $t \to t$ is C-provable. The theorem is a consequence of (12).

Let us consider X, L, and t. Let us observe that $t \to t$ is (SCI-prop-axioms L)-provable.

Let us consider C. Let t be a C-provable formula of L. Let us consider u. Observe that $(t \to u) \to u$ is C-provable.

Let us consider t. Let u be a C-provable formula of L. Let us consider v. Note that $(t \to (u \to v)) \to (t \to v)$ is C-provable.

Now we state the propositions:

(15) If $t \to (t \to u)$ is C-provable, then $t \to u$ is C-provable. The theorem is a consequence of (12).

(16) If $t \to (u \to v)$ is C-provable, then $u \to (t \to v)$ is C-provable. The theorem is a consequence of (12) and (13).

Let us consider X, L, t, and u. Let us observe that $(t \to (t \to u)) \to (t \to u)$ is (SCI-prop-axioms L)-provable.

Let us consider X, L, C, and t. Now we state the propositions:

(17) $\neg\neg t \to t$ is C-provable. The theorem is a consequence of (12).

(18) $t \to \neg\neg t$ is C-provable. The theorem is a consequence of (17).

Let us consider X, L, and t. Observe that $\neg\neg t \to t$ is (SCI-prop-axioms L)-provable and $t \to \neg\neg t$ is (SCI-prop-axioms L)-provable.

Let us consider u. Let us note that $(t \to u) \to (\neg u \to \neg t)$ is (SCI-prop-axioms L)-provable.

Let us consider X, L, C, t, and u. Now we state the propositions:

(19) If $\neg t \to u$ is C-provable, then $\neg u \to t$ is C-provable. The theorem is a consequence of (13).

(20) If $t \to \neg u$ is C-provable, then $u \to \neg t$ is C-provable. The theorem is a consequence of (13).

(21) $\neg t \to \neg u$ is C-provable if and only if $u \to t$ is C-provable. The theorem is a consequence of (13).

Let us consider X, L, C, and t. Let u be a C-provable formula of L. Observe that $\neg u \to t$ is C-provable and $t \to t$ is L-SCI-provable and $t \to \neg\neg t$ is L-SCI-provable and $\neg\neg t \to t$ is L-SCI-provable.

Let u be an L-SCI-provable formula of L. Observe that $t \to u$ is L-SCI-provable.

Now we state the proposition:

(22) $\neg t \to (t \to u)$ is C-provable.

Let us consider X, L, t, and u. One can verify that $\neg t \to (t \to u)$ is (SCI-prop-axioms L)-provable and $t \to (\neg t \to u)$ is (SCI-prop-axioms L)-provable.

Now we state the proposition:

(23) If $\neg t$ is C-provable, then $t \to u$ is C-provable.

Let us consider X, L, t, and u. Let us note that $t \to ((t \to u) \to u)$ is (SCI-prop-axioms L)-provable.

Now we state the proposition:

(24) $t \to (u \to v)$ is C-provable if and only if $t \to (\neg v \to \neg u)$ is C-provable. The theorem is a consequence of (12), (13), and (16).

Let us consider X, L, C, t, and u. Now we state the propositions:

(25) (i) $t \wedge u \to t$ is C-provable, and

(ii) $t \wedge u \to u$ is C-provable.

The theorem is a consequence of (19) and (13).

(26) $t \to (u \to t \wedge u)$ is C-provable. The theorem is a consequence of (21), (13), (16), and (24).

Let us consider X, L, t, and u. Observe that $t \wedge u \to t$ is (SCI-prop-axioms L)-provable and $t \wedge u \to u$ is (SCI-prop-axioms L)-provable and $t \to (u \to t \wedge u)$ is (SCI-prop-axioms L)-provable.

Let us consider C. Let u be a C-provable formula of L. One can verify that $t \to t \wedge u$ is C-provable and $t \to u \wedge t$ is C-provable.

Let t, u be C-provable formulae of L. Let us note that $t \wedge u$ is C-provable.

Now we state the propositions:

(27) $t \wedge u \to v$ is C-provable if and only if $t \to (u \to v)$ is C-provable. The theorem is a consequence of (12), (13), (16), and (15).

(28) $t \wedge u$ is C-provable if and only if t is C-provable and u is C-provable. The theorem is a consequence of (12).

(29) $t \to u \wedge v$ is C-provable if and only if $t \to u$ is C-provable and $t \to v$ is C-provable. The theorem is a consequence of (13), (16), and (12).

(30) (i) $t \to (t \vee u)$ is C-provable, and

(ii) $u \to (t \vee u)$ is C-provable.

The theorem is a consequence of (13).

Let us consider X, L, and t. Let us observe that $t \vee \neg t$ is (SCI-prop-axioms L)-provable.

Let us consider u. Note that $t \rightarrow (t \vee u)$ is (SCI-prop-axioms L)-provable and $u \rightarrow (t \vee u)$ is (SCI-prop-axioms L)-provable.

Let us consider C. Let t be a C-provable formula of L. One can check that $t \vee u$ is C-provable and $u \vee t$ is C-provable.

Now we state the propositions:

(31) $(\neg t \rightarrow t) \rightarrow t$ is C-provable. The theorem is a consequence of (12) and (13).

(32) $(t \vee u) \rightarrow v$ is C-provable if and only if $t \rightarrow v$ is C-provable and $u \rightarrow v$ is C-provable. The theorem is a consequence of (13), (21), and (31).

(33) Suppose $t \rightarrow v$ is C-provable and $u \rightarrow w$ is C-provable. Then

  (i) $(t \vee u) \rightarrow (v \vee w)$ is C-provable, and

  (ii) $t \wedge u \rightarrow v \wedge w$ is C-provable.

  The theorem is a consequence of (13), (32), and (29).

(34) $t \rightarrow u$ is C-provable if and only if u is $(C \cup \{t\})$-provable.
  Proof: Set $D = C \cup \{t\}$. If $t \rightarrow u$ is C-provable, then u is D-provable by [2, (6)], (12). □

From now on D denotes a (SCI-axioms L)-extending subset of L.

Let us consider X, L, and D. Let us note that every formula of L which is non D-provable is also non L-SCI-provable and there exists a formula of L which is D-provable.

Let us consider X, L, D, t, and u. Now we state the propositions:

(35) If $t \equiv u$ is D-provable, then $t \rightarrow u$ is D-provable. The theorem is a consequence of (12).

(36) If $t \equiv u$ is D-provable, then $\neg t \equiv \neg u$ is D-provable. The theorem is a consequence of (12).

(37) $(t \equiv u) \rightarrow (u \equiv t)$ is D-provable. The theorem is a consequence of (13), (16), and (12).

(38) If $t \equiv u$ is D-provable, then $u \equiv t$ is D-provable. The theorem is a consequence of (37) and (12).

Now we state the propositions:

(39) $(t \equiv u) \wedge (v \equiv u) \rightarrow (t \equiv v)$ is D-provable. The theorem is a consequence of (37), (13), (16), and (12).

(40) If t is D-provable and $t \Rightarrow u$ is D-provable, then u is D-provable. The theorem is a consequence of (35), (12), and (28).

(41) Suppose $t \equiv u$ is D-provable and $v \equiv w$ is D-provable. Then

(i) $t \wedge v \equiv u \wedge w$ is D-provable, and
(ii) $t \wedge v \rightarrow u \wedge w$ is D-provable, and
(iii) if $t \wedge v$ is D-provable, then $u \wedge w$ is D-provable, and
(iv) $(t \vee v) \equiv (u \vee w)$ is D-provable, and
(v) $(t \vee v) \rightarrow (u \vee w)$ is D-provable, and
(vi) if $t \vee v$ is D-provable, then $u \vee w$ is D-provable, and
(vii) $(t \rightarrow v) \equiv (u \rightarrow w)$ is D-provable, and
(viii) $(t \rightarrow v) \rightarrow (u \rightarrow w)$ is D-provable, and
(ix) if $t \rightarrow v$ is D-provable, then $u \rightarrow w$ is D-provable, and
(x) $(t \leftrightarrow v) \equiv (u \leftrightarrow w)$ is D-provable, and
(xi) $(t \leftrightarrow v) \rightarrow (u \leftrightarrow w)$ is D-provable, and
(xii) if $t \leftrightarrow v$ is D-provable, then $u \leftrightarrow w$ is D-provable, and
(xiii) $(t \equiv v) \equiv (u \equiv w)$ is D-provable, and
(xiv) $(t \equiv v) \rightarrow (u \equiv w)$ is D-provable, and
(xv) if $t \equiv v$ is D-provable, then $u \equiv w$ is D-provable, and
(xvi) $(t \Rightarrow v) \equiv (u \Rightarrow w)$ is D-provable, and
(xvii) $(t \Rightarrow v) \rightarrow (u \Rightarrow w)$ is D-provable, and
(xviii) if $t \Rightarrow v$ is D-provable, then $u \Rightarrow w$ is D-provable.

The theorem is a consequence of (12) and (35).

Let us consider X, L, D, t, u, and v. Now we state the propositions:

(42) (i) $(t \equiv u) \rightarrow (t \wedge v \equiv u \wedge v)$ is D-provable, and
(ii) $(t \equiv u) \rightarrow (v \wedge t \equiv v \wedge u)$ is D-provable, and
(iii) $(t \equiv u) \rightarrow ((t \vee v) \equiv (u \vee v))$ is D-provable, and
(iv) $(t \equiv u) \rightarrow ((v \vee t) \equiv (v \vee u))$ is D-provable, and
(v) $(t \equiv u) \rightarrow ((t \rightarrow v) \equiv (u \rightarrow v))$ is D-provable, and
(vi) $(t \equiv u) \rightarrow ((v \rightarrow t) \equiv (v \rightarrow u))$ is D-provable, and
(vii) $(t \equiv u) \rightarrow ((t \leftrightarrow v) \equiv (u \leftrightarrow v))$ is D-provable, and
(viii) $(t \equiv u) \rightarrow ((v \leftrightarrow t) \equiv (v \leftrightarrow u))$ is D-provable, and
(ix) $(t \equiv u) \rightarrow ((t \equiv v) \equiv (u \equiv v))$ is D-provable, and
(x) $(t \equiv u) \rightarrow ((v \equiv t) \equiv (v \equiv u))$ is D-provable, and
(xi) $(t \equiv u) \rightarrow ((t \Rightarrow v) \equiv (u \Rightarrow v))$ is D-provable, and
(xii) $(t \equiv u) \rightarrow ((v \Rightarrow t) \equiv (v \Rightarrow u))$ is D-provable.

The theorem is a consequence of (13) and (29).

(43) Suppose $t \equiv u$ is D-provable. Then

(i) t is D-provable iff u is D-provable, and

(ii) $t \wedge v \equiv u \wedge v$ is D-provable, and

(iii) $t \wedge v \to u \wedge v$ is D-provable, and

(iv) if $t \wedge v$ is D-provable, then $u \wedge v$ is D-provable, and

(v) $v \wedge t \equiv v \wedge u$ is D-provable, and

(vi) $v \wedge t \to v \wedge u$ is D-provable, and

(vii) if $v \wedge t$ is D-provable, then $v \wedge u$ is D-provable, and

(viii) $(t \vee v) \equiv (u \vee v)$ is D-provable, and

(ix) $(t \vee v) \to (u \vee v)$ is D-provable, and

(x) if $t \vee v$ is D-provable, then $u \vee v$ is D-provable, and

(xi) $(v \vee t) \equiv (v \vee u)$ is D-provable, and

(xii) $(v \vee t) \to (v \vee u)$ is D-provable, and

(xiii) if $v \vee t$ is D-provable, then $v \vee u$ is D-provable, and

(xiv) $(t \to v) \equiv (u \to v)$ is D-provable, and

(xv) $(t \to v) \to (u \to v)$ is D-provable, and

(xvi) if $t \to v$ is D-provable, then $u \to v$ is D-provable, and

(xvii) $(v \to t) \equiv (v \to u)$ is D-provable, and

(xviii) $(v \to t) \to (v \to u)$ is D-provable, and

(xix) if $v \to t$ is D-provable, then $v \to u$ is D-provable, and

(xx) $(t \leftrightarrow v) \equiv (u \leftrightarrow v)$ is D-provable, and

(xxi) $(t \leftrightarrow v) \to (u \leftrightarrow v)$ is D-provable, and

(xxii) if $t \leftrightarrow v$ is D-provable, then $u \leftrightarrow v$ is D-provable, and

(xxiii) $(v \leftrightarrow t) \equiv (v \leftrightarrow u)$ is D-provable, and

(xxiv) $(v \leftrightarrow t) \to (v \leftrightarrow u)$ is D-provable, and

(xxv) if $v \leftrightarrow t$ is D-provable, then $v \leftrightarrow u$ is D-provable, and

(xxvi) $(t \equiv v) \equiv (u \equiv v)$ is D-provable, and

(xxvii) $(t \equiv v) \to (u \equiv v)$ is D-provable, and

(xxviii) if $t \equiv v$ is D-provable, then $u \equiv v$ is D-provable, and

(xxix) $(v \equiv t) \equiv (v \equiv u)$ is D-provable, and

(xxx) $(v \equiv t) \to (v \equiv u)$ is D-provable, and

(xxxi) if $v \equiv t$ is D-provable, then $v \equiv u$ is D-provable, and

(xxxii) $(t \Rightarrow v) \equiv (u \Rightarrow v)$ is D-provable, and

(xxxiii) $(t \Rightarrow v) \rightarrow (u \Rightarrow v)$ is D-provable, and

(xxxiv) if $t \Rightarrow v$ is D-provable, then $u \Rightarrow v$ is D-provable, and

(xxxv) $(v \Rightarrow t) \equiv (v \Rightarrow u)$ is D-provable, and

(xxxvi) $(v \Rightarrow t) \rightarrow (v \Rightarrow u)$ is D-provable, and

(xxxvii) if $v \Rightarrow t$ is D-provable, then $v \Rightarrow u$ is D-provable.

The theorem is a consequence of (38), (35), (12), and (42).

4. Congruences

Let us consider X and L.

A congruence of L is an equivalence relation of L defined by

(Def. 47) for every t, u, v, and w such that $\langle t, u\rangle$, $\langle v, w\rangle \in it$ holds $\langle \neg t, \neg u\rangle$, $\langle t \wedge v, u \wedge w\rangle$, $\langle t \vee v, u \vee w\rangle$, $\langle t \rightarrow v, u \rightarrow w\rangle$, $\langle t \leftrightarrow v, u \leftrightarrow w\rangle$, $\langle t \equiv v, u \equiv w\rangle \in it$.

In the sequel E denotes a congruence of L.

Let us consider X and L. One can verify that there exists a family of subsets of L which is non empty.

Let us consider E.

An equivalence class of E is an element of Classes E. Let us consider t. The functor E-class t yielding an equivalence class of E is defined by the term

(Def. 48) $[t]_E$.

Now we state the proposition:

(44) $\langle t, u\rangle \in E$ if and only if E-class $t = E$-class u.

PROOF: If $\langle t, u\rangle \in E$, then E-class $t = E$-class u by [3, (18), (23)]. $\square$

From now on d, e denote equivalence classs of E.

Now we state the proposition:

(45) There exists t such that $d = E$-class t.

Let us consider X, L, E, and d. The functor $\neg d$ yielding an equivalence class of E is defined by

(Def. 49) there exists t such that $d = E$-class t and $it = E$-class $\neg t$.

Let us consider e. The functors: $d \wedge e$, $d \vee e$, $d \rightarrow e$, and $d \leftrightarrow e$ yielding equivalence classs of E are defined by conditions

(Def. 50) there exists t and there exists u such that $d = E$-class t and $e = E$-class u and $d \wedge e = E$-class $t \wedge u$,

(Def. 51) there exists t and there exists u such that $d = E$-class t and $e = E$-class u and $d \vee e = E$-class$(t \vee u)$, (Def. 52) there exists t and there exists u such that $d = E\text{-class}\,t$ and $e = E\text{-class}\,u$ and $d \rightarrow e = E\text{-class}(t \rightarrow u)$,

(Def. 53) there exists t and there exists u such that $d = E\text{-class}\,t$ and $e = E\text{-class}\,u$ and $d \leftrightarrow e = E\text{-class}(t \leftrightarrow u)$,

respectively. Let us consider D. The functor $\mathrm{EqRel}(D)$ yielding a congruence of L is defined by

(Def. 54) for every t and u, $\langle t, u\rangle \in it$ iff $t \equiv u$ is D-provable.

An equivalence class of D is an equivalence class of $\mathrm{EqRel}(D)$. Let us consider t. The functor $D\text{-class}\,t$ yielding an equivalence class of D is defined by the term

(Def. 55) $\mathrm{EqRel}(D)\text{-class}\,t$.

Now we state the proposition:

(46) $t \equiv u$ is D-provable if and only if $D\text{-class}\,t = D\text{-class}\,u$. The theorem is a consequence of (44).

In the sequel x, y, z denote equivalence classs of D.

Now we state the proposition:

(47) There exists t such that $x = D\text{-class}\,t$. The theorem is a consequence of (45).

Let us consider X, L, D, and x. We say that x is D-provable if and only if

(Def. 56) there exists t such that $x = D\text{-class}\,t$ and t is D-provable.

Now we state the proposition:

(48) $y = \neg x$ if and only if there exists t such that $x = D\text{-class}\,t$ and $y = D\text{-class}\,\neg t$.

Let us consider X, L, and D. Let t be a D-provable formula of L. Let us note that $D\text{-class}\,t$ is D-provable and there exists an equivalence class of D which is D-provable.

Now we state the proposition:

(49) If $D\text{-class}\,t$ is D-provable, then t is D-provable. The theorem is a consequence of (46), (35), and (12).

Let us consider X, L, and D. Let x be a D-provable equivalence class of D. One can verify that every element of x is D-provable.

Let us consider x and y. Now we state the propositions:

(50) $x \wedge y$ is D-provable if and only if x is D-provable and y is D-provable. The theorem is a consequence of (47), (49), and (28).

(51) $x \equiv y$ is D-provable if and only if $x = y$. The theorem is a consequence of (47), (46), and (49).

Now we state the propositions:

(52) (i) D-class $\neg t = \neg(D\text{-class}\, t)$, and

(ii) D-class $t \wedge u = (D\text{-class}\, t) \wedge (D\text{-class}\, u)$, and

(iii) $D\text{-class}(t \vee u) = (D\text{-class}\, t) \vee (D\text{-class}\, u)$, and

(iv) $D\text{-class}(t \rightarrow u) = (D\text{-class}\, t) \rightarrow (D\text{-class}\, u)$, and

(v) $D\text{-class}(t \leftrightarrow u) = (D\text{-class}\, t) \leftrightarrow (D\text{-class}\, u)$, and

(vi) $D\text{-class}(t \equiv u) = (D\text{-class}\, t) \equiv (D\text{-class}\, u)$.

(53) If x is D-provable, then $x \vee y$ is D-provable and $y \vee x$ is D-provable. The theorem is a consequence of (47) and (49).

Let us consider X, L, D, t, and u. Now we state the propositions:

(54) $t \leftrightarrow u$ is D-provable if and only if $t \rightarrow u$ is D-provable and $u \rightarrow t$ is D-provable. The theorem is a consequence of (12) and (28).

(55) If $t \leftrightarrow u$ is D-provable, then $u \leftrightarrow t$ is D-provable. The theorem is a consequence of (54).

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