

# Suszko's Non-Fregean Logics. Part I<sup>1</sup>

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**Summary.** The basic properties of non-Fregean logics in general and of Sentential Calculus with Identity in particular, as introduced by Roman Suszko in [4] and [5].

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## 1. PRELIMINARIES

From now on  $k, m, n$  denote elements of  $\mathbb{N}$ ,  $i, j$  denote natural numbers,  $a, b, c$  denote objects,  $y, z$  denote sets, and  $p, q, r, s$  denote finite sequences.

The functor **VARS** yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all  $\langle 0, k \rangle$  where  $k$  is an element of  $\mathbb{N}$ .

Observe that **VARS** is non empty and antichain-like.

A variable is an element of **VARS**. The functors: 'not', **' $\wedge$ '**, and **' $\vee$ '** yielding finite sequences are defined by terms

(Def. 2)  $\langle 11 \rangle$ ,

(Def. 3)  $\langle 21 \rangle$ ,

(Def. 4)  $\langle 22 \rangle$ ,

respectively. The functors: **' $\rightarrow$ '**, **' $\leftrightarrow$ '**, and **' $\equiv$ '** yielding finite sequences are defined by terms

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(Def. 5)  $\langle 23 \rangle$ ,

(Def. 6)  $\langle 24 \rangle$ ,

(Def. 7)  $\langle 25 \rangle$ ,

respectively. The functors: **SCI-unops** and **SCI-binops** yielding non empty, finite sequence-membered sets are defined by terms

(Def. 8)  $\{\text{'not'}\}$ ,

(Def. 9)  $\{\text{'}\wedge\text{'}, \text{'}\vee\text{'}, \text{'}\rightarrow\text{'}, \text{'}\leftrightarrow\text{'}, \text{'}\equiv\text{'}\}$ ,

respectively. Now we state the proposition:

(1) (i)  $a \in \text{SCI-unops}$  iff  $a = \text{'not'}$ , and

(ii)  $a \in \text{SCI-binops}$  iff  $a = \text{'}\wedge\text{'}$  or  $a = \text{'}\vee\text{'}$  or  $a = \text{'}\rightarrow\text{'}$  or  $a = \text{'}\leftrightarrow\text{'}$  or  $a = \text{'}\equiv\text{'}$ .

Let  $F, G$  be non empty, finite sequence-membered sets. One can verify that  $F \cup G$  is non empty and finite sequence-membered.

The functor **SCI-ops** yielding a non empty, finite sequence-membered set is defined by the term

(Def. 10)  $\text{SCI-unops} \cup \text{SCI-binops}$ .

Now we state the proposition:

(2) (i) if  $p = \text{'not'}$ , then  $p(1) = 11$ , and

(ii) if  $p = \text{'}\wedge\text{'}$ , then  $p(1) = 21$ , and

(iii) if  $p = \text{'}\vee\text{'}$ , then  $p(1) = 22$ , and

(iv) if  $p = \text{'}\rightarrow\text{'}$ , then  $p(1) = 23$ , and

(v) if  $p = \text{'}\leftrightarrow\text{'}$ , then  $p(1) = 24$ , and

(vi) if  $p = \text{'}\equiv\text{'}$ , then  $p(1) = 25$ .

One can verify that **SCI-ops** is non empty and antichain-like.

The functor **SCI-symbols** yielding a non empty, finite sequence-membered set is defined by the term

(Def. 11)  $\text{VARs} \cup \text{SCI-ops}$ .

The functors: **VARs**, **SCI-ops**, **SCI-unops**, and **SCI-binops** yield non empty subsets of **SCI-symbols**. The functors:  $\text{'not'}$ ,  $\text{'}\wedge\text{'}$ ,  $\text{'}\vee\text{'}$ ,  $\text{'}\rightarrow\text{'}$ , and  $\text{'}\leftrightarrow\text{'}$  yield elements of **SCI-symbols**. Observe that **SCI-symbols** is non trivial and antichain-like.

Note that the functor **SCI-symbols** yields a non trivial Polish language. The functor **SCI-op-arity** yielding a function from **SCI-ops** into  $\mathbb{N}$  is defined by the term

(Def. 12)  $(\text{SCI-binops} \mapsto 2) + \cdot (\text{SCI-unops} \mapsto 1)$ .

The functor **SCI-arity** yielding a Polish arity-function of **SCI-symbols** is defined by the term

(Def. 13) SCI-op-arity  $\vdash \cdot (\text{VARS} \mapsto 0)$ .

Now we state the propositions:

(3) If  $a \in \text{VARS}$ , then  $(\text{SCI-arity})(a) = 0$ .

(4) (i)  $(\text{SCI-arity})('not') = 1$ , and

(ii) for every  $a$  such that  $a \in \text{SCI-binops}$  holds  $(\text{SCI-arity})(a) = 2$ .

(5) The Polish atoms( $\text{SCI-symbols}$ ,  $\text{SCI-arity}$ ) =  $\text{VARS}$ . The theorem is a consequence of (4) and (3).

The functor **SCI-formula-set** yielding a full Polish language of SCI-symbols is defined by the term

(Def. 14) Polish-WFF-set( $\text{SCI-symbols}$ ,  $\text{SCI-arity}$ ).

**An SCI-formula** is a Polish WFF of SCI-symbols and SCI-arity. Let us observe that there exists a subset of SCI-formula-set which is non empty.

Let us consider  $n$ . The functor  $x_n$  yielding an SCI-formula is defined by the term

(Def. 15)  $\langle 0, n \rangle$ .

In the sequel  $X$  denotes an extension of SCI-arity,  $L$  denotes a Polish-ext-set of  $X$ , and  $t, u, v, w$  denote formulae of  $L$ .

Let us consider  $X$ . Now we state the propositions:

(6)  $\text{SCI-symbols} \subseteq \text{dom } X$ .

(7) (i)  $'not' \in \text{dom } X$ , and

(ii)  $X('not') = 1$ , and

(iii) for every  $a$  such that  $a \in \text{SCI-binops}$  holds  $a \in \text{dom } X$  and  $X(a) = 2$ .

The theorem is a consequence of (6) and (4).

Let us consider  $X$ ,  $L$ , and  $n$ . The functor  $x.(n, L)$  yielding a formula of  $L$  is defined by the term

(Def. 16)  $x_n$ .

Now we state the proposition:

(8) If  $m \neq n$ , then  $x_m \neq x_n$ .

Let us consider  $p$ . The functor  $\neg p$  yielding a finite sequence is defined by the term

(Def. 17)  $'not' \frown p$ .

Let us consider  $q$ . The functors:  $p \wedge q$ ,  $p \vee q$ ,  **$p \rightarrow q$** ,  **$p \leftrightarrow q$** , and  **$p \equiv q$**  yielding finite sequences are defined by terms

(Def. 18)  $'\wedge' \frown (p \frown q)$ ,

(Def. 19)  $'\vee' \frown (p \frown q)$ ,

(Def. 20)  $'\rightarrow' \frown (p \frown q)$ ,

(Def. 21)  $'\leftrightarrow' \wedge (p \wedge q)$ ,

(Def. 22)  $'\equiv' \wedge (p \wedge q)$ ,

respectively. Let us consider  $X$ ,  $L$ , and  $t$ . One can check that the functor  $\neg t$  is defined by the term

(Def. 23)  $(\text{Polish-unOp}(X, L, 'not'))(t)$ .

Let us consider  $u$ . The functors:  $t \wedge u$ ,  $t \vee u$ ,  $t \rightarrow u$ ,  $t \leftrightarrow u$ , and  $t \equiv u$  are defined by terms

(Def. 24)  $(\text{Polish-binOp}(X, L, '\wedge'))(t, u)$ ,

(Def. 25)  $(\text{Polish-binOp}(X, L, '\vee'))(t, u)$ ,

(Def. 26)  $(\text{Polish-binOp}(X, L, '\rightarrow'))(t, u)$ ,

(Def. 27)  $(\text{Polish-binOp}(X, L, '\leftrightarrow'))(t, u)$ ,

(Def. 28)  $(\text{Polish-binOp}(X, L, '\equiv'))(t, u)$ ,

respectively. Note that the functor  $\neg t$  yields a formula of  $L$ . Let us consider  $u$ . The functors:  $t \wedge u$ ,  $t \vee u$ ,  $t \rightarrow u$ ,  $t \leftrightarrow u$ , and  $t \equiv u$  yield formulae of  $L$ . The functor  $t \Rightarrow u$  yielding a formula of  $L$  is defined by the term

(Def. 29)  $t \equiv t \wedge u$ .

Let  $u$  be an SCI-formula. The functors:  $t \rightarrow u$  and  $t \equiv u$  yield formulae of  $L$ . The functors:  $u \rightarrow t$  and  $u \equiv t$  yield formulae of  $L$ . We say that  $t$  is atomic if and only if

(Def. 30)  $t \in \text{the Polish atoms}(\text{SCI-symbols}, \text{SCI-arity})$ .

We say that  $t$  is negative if and only if

(Def. 31)  $\text{PolishExtHead}(t) = 'not'$ .

We say that  $t$  is conjunctive if and only if

(Def. 32)  $\text{PolishExtHead}(t) = '\wedge'$ .

We say that  $t$  is disjunctive if and only if

(Def. 33)  $\text{PolishExtHead}(t) = '\vee'$ .

We say that  $t$  is conditional if and only if

(Def. 34)  $\text{PolishExtHead}(t) = '\rightarrow'$ .

We say that  $t$  is biconditional if and only if

(Def. 35)  $\text{PolishExtHead}(t) = '\leftrightarrow'$ .

We say that  $t$  is an equality if and only if

(Def. 36)  $\text{PolishExtHead}(t) = '\equiv'$ .

Let us consider  $t$ . Now we state the propositions:

(9)  $t$  is atomic if and only if  $t \in \text{VARS}$ .

(10)  $t$  is negative if and only if there exists  $u$  such that  $t = \neg u$ .

PROOF: ' $\text{not}' \in \text{dom } X$  and  $X('not') = 1$ . If  $t$  is negative, then there exists  $u$  such that  $t = \neg u$  by [1, (9)].  $\square$

(11)  $t$  is conjunctive if and only if there exists  $u$  and there exists  $v$  such that  $t = u \wedge v$ .

PROOF: ' $\wedge' \in \text{dom } X$  and  $X(' \wedge ') = 2$ . If  $t$  is conjunctive, then there exists  $u$  and there exists  $v$  such that  $t = u \wedge v$  by [1, (11)].  $\square$

## 2. SCI AXIOMS

Let us consider  $X$  and  $L$ . The functors: **SCI-prop-axioms  $L$**  and **SCI-id-axioms  $L$**  yielding non empty subsets of  $L$  are defined by conditions

(Def. 37) for every  $a$ ,  $a \in \text{SCI-prop-axioms } L$  iff there exists  $t$  and there exists  $u$  and there exists  $v$  such that  $a = t \rightarrow (u \rightarrow t)$  or  $a = (t \rightarrow (u \rightarrow v)) \rightarrow ((t \rightarrow u) \rightarrow (t \rightarrow v))$  or  $a = (\neg t \rightarrow \neg u) \rightarrow (u \rightarrow t)$  or  $a = t \wedge u \rightarrow \neg(t \rightarrow \neg u)$  or  $a = \neg(t \rightarrow \neg u) \rightarrow t \wedge u$  or  $a = (t \vee u) \rightarrow (\neg t \rightarrow u)$  or  $a = (\neg t \rightarrow u) \rightarrow (t \vee u)$  or  $a = (t \leftrightarrow u) \rightarrow (t \rightarrow u) \wedge (u \rightarrow t)$  or  $a = (t \rightarrow u) \wedge (u \rightarrow t) \rightarrow (t \leftrightarrow u)$ ,

(Def. 38) for every  $a$ ,  $a \in \text{SCI-id-axioms } L$  iff there exists  $t$  and there exists  $u$  and there exists  $v$  and there exists  $w$  such that  $a = t \equiv t$  or  $a = (t \equiv u) \rightarrow (\neg t \equiv \neg u)$  or  $a = (t \equiv u) \wedge (v \equiv w) \rightarrow (t \wedge v \equiv u \wedge w)$  or  $a = (t \equiv u) \wedge (v \equiv w) \rightarrow ((t \vee v) \equiv (u \vee w))$  or  $a = (t \equiv u) \wedge (v \equiv w) \rightarrow ((t \rightarrow v) \equiv (u \rightarrow w))$  or  $a = (t \equiv u) \wedge (v \equiv w) \rightarrow ((t \leftrightarrow v) \equiv (u \leftrightarrow w))$  or  $a = (t \equiv u) \wedge (v \equiv w) \rightarrow ((t \equiv v) \equiv (u \equiv w))$  or  $a = (t \equiv u) \rightarrow (t \rightarrow u)$ ,

respectively. Let  $B$  be a subset of  $L$ . Observe that there exists a non empty subset of  $L$  which is  $B$ -extending.

The functor **SCI-axioms  $L$**  yielding a (SCI-prop-axioms  $L$ )-extending subset of  $L$  is defined by the term

(Def. 39) **SCI-prop-axioms  $L \cup \text{SCI-id-axioms } L$ .**

From now on  $R, R_1, R_2$  denote rules of  $L$ .

Let us consider  $X$  and  $L$ . The functor **SCI-MP  $L$**  yielding a rule of  $L$  is defined by the term

(Def. 40) the set of all  $\{\{t, t \rightarrow u\}, u\}$  where  $t, u$  are formulae of  $L$ .

The functor **SCI-rules  $L$**  yielding a rule of  $L$  is defined by the term

(Def. 41) **SCI-MP  $L$ .**

A formula-sequence of  $L$  is a finite sequence of elements of  $L$ .

A formula-finset of  $L$  is a finite subset of  $L$ . In the sequel  $A, A_1, A_2$  denote non empty subsets of  $L$ ,  $B, B_1, B_2$  denote subsets of  $L$ ,  $P, P_1, P_2$  denote formula-sequences of  $L$ , and  $S, S_1, S_2$  denote formula-finsets of  $L$ .

Let us consider  $X$ ,  $L$ , and  $t$ . One can verify that the functor  $\{t\}$  yields a formula-finset of  $L$ .

### 3. PROVABILITY

Let us consider  $X$ ,  $L$ ,  $B$ , and  $a$ . We say that  $a$  is  $B$ -provable if and only if

(Def. 42)  $a$  is  $(B, (\text{SCI-rules } L))$ -provable.

Observe that every object which is  $B$ -axiomatic is also  $B$ -provable.

Now we state the proposition:

(12) If  $t$  is  $B$ -provable and  $t \rightarrow u$  is  $B$ -provable, then  $u$  is  $B$ -provable.

Let us consider  $X$ ,  $L$ , and  $a$ . We say that  $a$  is  $L$ -prop-axiomatic if and only if

(Def. 43)  $a$  is  $(\text{SCI-prop-axioms } L)$ -axiomatic.

We say that  $a$  is  $L$ -id-axiomatic if and only if

(Def. 44)  $a$  is  $(\text{SCI-id-axioms } L)$ -axiomatic.

We say that  $a$  is  $L$ -SCI-axiomatic if and only if

(Def. 45)  $a$  is  $(\text{SCI-axioms } L)$ -axiomatic.

We say that  $a$  is  $L$ -SCI-provable if and only if

(Def. 46)  $a$  is  $(\text{SCI-axioms } L)$ -provable.

One can verify that every element of  $\text{SCI-prop-axioms } L$  is  $L$ -prop-axiomatic and every element of  $\text{SCI-id-axioms } L$  is  $L$ -id-axiomatic and every element of  $\text{SCI-axioms } L$  is  $L$ -SCI-axiomatic and every object which is  $L$ -SCI-axiomatic is also  $L$ -SCI-provable and every object which is  $L$ -prop-axiomatic is also  $L$ -SCI-axiomatic and every object which is  $L$ -id-axiomatic is also  $L$ -SCI-axiomatic.

Let us consider  $t$ . Observe that  $t \equiv t$  is  $L$ -id-axiomatic.

Let us consider  $u$ . One can verify that  $t \rightarrow (u \rightarrow t)$  is  $L$ -prop-axiomatic and  $(\neg t \rightarrow \neg u) \rightarrow (u \rightarrow t)$  is  $L$ -prop-axiomatic and  $t \wedge u \rightarrow \neg(t \rightarrow \neg u)$  is  $L$ -prop-axiomatic and  $\neg(t \rightarrow \neg u) \rightarrow t \wedge u$  is  $L$ -prop-axiomatic and  $(t \vee u) \rightarrow (\neg t \rightarrow u)$  is  $L$ -prop-axiomatic and  $(\neg t \rightarrow u) \rightarrow (t \vee u)$  is  $L$ -prop-axiomatic and  $(t \leftrightarrow u) \rightarrow (t \rightarrow u) \wedge (u \rightarrow t)$  is  $L$ -prop-axiomatic and  $(t \rightarrow u) \wedge (u \rightarrow t) \rightarrow (t \leftrightarrow u)$  is  $L$ -prop-axiomatic and  $(t \equiv u) \rightarrow (\neg t \equiv \neg u)$  is  $L$ -id-axiomatic and  $(t \equiv u) \rightarrow (t \rightarrow u)$  is  $L$ -id-axiomatic.

Let us consider  $v$ . Observe that  $(t \rightarrow (u \rightarrow v)) \rightarrow ((t \rightarrow u) \rightarrow (t \rightarrow v))$  is  $L$ -prop-axiomatic.

Let us consider  $w$ . Let  $O$  be an element of  $\text{SCI-binops}$ . Note that  $(t \equiv u) \wedge (v \equiv w) \rightarrow ((\text{Polish-binOp}(X, L, O))(t, v) \equiv (\text{Polish-binOp}(X, L, O))(u, w))$  is  $L$ -id-axiomatic and there exists a formula of  $L$  which is  $L$ -prop-axiomatic and there exists a formula of  $L$  which is  $L$ -id-axiomatic.

In the sequel  $C$  denotes a  $(\text{SCI-prop-axioms } L)$ -extending subset of  $L$ .

Let us consider  $X$  and  $L$ . Let us note that every formula of  $L$  which is  $L$ -prop-axiomatic is also (SCI-prop-axioms  $L$ )-provable.

Let us consider  $C$ . Let us note that every formula of  $L$  which is non  $C$ -provable is also non (SCI-prop-axioms  $L$ )-provable and there exists a formula of  $L$  which is  $C$ -provable.

Let us consider  $t$ . Let  $u$  be a  $C$ -provable formula of  $L$ . Observe that  $t \rightarrow u$  is  $C$ -provable.

Now we state the propositions:

- (13) If  $t \rightarrow u$  is  $C$ -provable and  $u \rightarrow v$  is  $C$ -provable, then  $t \rightarrow v$  is  $C$ -provable.

The theorem is a consequence of (12).

- (14)  $t \rightarrow t$  is  $C$ -provable. The theorem is a consequence of (12).

Let us consider  $X$ ,  $L$ , and  $t$ . Let us observe that  $t \rightarrow t$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $C$ . Let  $t$  be a  $C$ -provable formula of  $L$ . Let us consider  $u$ . Observe that  $(t \rightarrow u) \rightarrow u$  is  $C$ -provable.

Let us consider  $t$ . Let  $u$  be a  $C$ -provable formula of  $L$ . Let us consider  $v$ . Note that  $(t \rightarrow (u \rightarrow v)) \rightarrow (t \rightarrow v)$  is  $C$ -provable.

Now we state the propositions:

- (15) If  $t \rightarrow (t \rightarrow u)$  is  $C$ -provable, then  $t \rightarrow u$  is  $C$ -provable. The theorem is a consequence of (12).

- (16) If  $t \rightarrow (u \rightarrow v)$  is  $C$ -provable, then  $u \rightarrow (t \rightarrow v)$  is  $C$ -provable. The theorem is a consequence of (12) and (13).

Let us consider  $X$ ,  $L$ ,  $t$ , and  $u$ . Let us observe that  $(t \rightarrow (t \rightarrow u)) \rightarrow (t \rightarrow u)$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $X$ ,  $L$ ,  $C$ , and  $t$ . Now we state the propositions:

- (17)  $\neg\neg t \rightarrow t$  is  $C$ -provable. The theorem is a consequence of (12).

- (18)  $t \rightarrow \neg\neg t$  is  $C$ -provable. The theorem is a consequence of (17).

Let us consider  $X$ ,  $L$ , and  $t$ . Observe that  $\neg\neg t \rightarrow t$  is (SCI-prop-axioms  $L$ )-provable and  $t \rightarrow \neg\neg t$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $u$ . Let us note that  $(t \rightarrow u) \rightarrow (\neg u \rightarrow \neg t)$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $X$ ,  $L$ ,  $C$ ,  $t$ , and  $u$ . Now we state the propositions:

- (19) If  $\neg t \rightarrow u$  is  $C$ -provable, then  $\neg u \rightarrow t$  is  $C$ -provable. The theorem is a consequence of (13).

- (20) If  $t \rightarrow \neg u$  is  $C$ -provable, then  $u \rightarrow \neg t$  is  $C$ -provable. The theorem is a consequence of (13).

- (21)  $\neg t \rightarrow \neg u$  is  $C$ -provable if and only if  $u \rightarrow t$  is  $C$ -provable. The theorem is a consequence of (13).

Let us consider  $X$ ,  $L$ ,  $C$ , and  $t$ . Let  $u$  be a  $C$ -provable formula of  $L$ . Observe that  $\neg u \rightarrow t$  is  $C$ -provable and  $t \rightarrow t$  is  $L$ -SCI-provable and  $t \rightarrow \neg\neg t$  is  $L$ -SCI-provable and  $\neg\neg t \rightarrow t$  is  $L$ -SCI-provable.

Let  $u$  be an  $L$ -SCI-provable formula of  $L$ . Observe that  $t \rightarrow u$  is  $L$ -SCI-provable.

Now we state the proposition:

(22)  $\neg t \rightarrow (t \rightarrow u)$  is  $C$ -provable.

Let us consider  $X$ ,  $L$ ,  $t$ , and  $u$ . One can verify that  $\neg t \rightarrow (t \rightarrow u)$  is (SCI-prop-axioms  $L$ )-provable and  $t \rightarrow (\neg t \rightarrow u)$  is (SCI-prop-axioms  $L$ )-provable.

Now we state the proposition:

(23) If  $\neg t$  is  $C$ -provable, then  $t \rightarrow u$  is  $C$ -provable.

Let us consider  $X$ ,  $L$ ,  $t$ , and  $u$ . Let us note that  $t \rightarrow ((t \rightarrow u) \rightarrow u)$  is (SCI-prop-axioms  $L$ ) provable.

Now we state the proposition:

(24)  $t \rightarrow (u \rightarrow v)$  is  $C$ -provable if and only if  $t \rightarrow (\neg v \rightarrow \neg u)$  is  $C$ -provable. The theorem is a consequence of (12), (13), and (16).

Let us consider  $X$ ,  $L$ ,  $C$ ,  $t$ , and  $u$ . Now we state the propositions:

(25) (i)  $t \wedge u \rightarrow t$  is  $C$ -provable, and

(ii)  $t \wedge u \rightarrow u$  is  $C$ -provable.

The theorem is a consequence of (19) and (13).

(26)  $t \rightarrow (u \rightarrow t \wedge u)$  is  $C$ -provable. The theorem is a consequence of (21), (13), (16), and (24).

Let us consider  $X$ ,  $L$ ,  $t$ , and  $u$ . Observe that  $t \wedge u \rightarrow t$  is (SCI-prop-axioms  $L$ )-provable and  $t \wedge u \rightarrow u$  is (SCI-prop-axioms  $L$ )-provable and  $t \rightarrow (u \rightarrow t \wedge u)$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $C$ . Let  $u$  be a  $C$ -provable formula of  $L$ . One can verify that  $t \rightarrow t \wedge u$  is  $C$ -provable and  $t \rightarrow u \wedge t$  is  $C$ -provable.

Let  $t$ ,  $u$  be  $C$ -provable formulae of  $L$ . Let us note that  $t \wedge u$  is  $C$ -provable.

Now we state the propositions:

(27)  $t \wedge u \rightarrow v$  is  $C$ -provable if and only if  $t \rightarrow (u \rightarrow v)$  is  $C$ -provable. The theorem is a consequence of (12), (13), (16), and (15).

(28)  $t \wedge u$  is  $C$ -provable if and only if  $t$  is  $C$ -provable and  $u$  is  $C$ -provable. The theorem is a consequence of (12).

(29)  $t \rightarrow u \wedge v$  is  $C$ -provable if and only if  $t \rightarrow u$  is  $C$ -provable and  $t \rightarrow v$  is  $C$ -provable. The theorem is a consequence of (13), (16), and (12).

(30) (i)  $t \rightarrow (t \vee u)$  is  $C$ -provable, and

(ii)  $u \rightarrow (t \vee u)$  is  $C$ -provable.

The theorem is a consequence of (13).

Let us consider  $X$ ,  $L$ , and  $t$ . Let us observe that  $t \vee \neg t$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $u$ . Note that  $t \rightarrow (t \vee u)$  is (SCI-prop-axioms  $L$ )-provable and  $u \rightarrow (t \vee u)$  is (SCI-prop-axioms  $L$ )-provable.

Let us consider  $C$ . Let  $t$  be a  $C$ -provable formula of  $L$ . One can check that  $t \vee u$  is  $C$ -provable and  $u \vee t$  is  $C$ -provable.

Now we state the propositions:

- (31)  $(\neg t \rightarrow t) \rightarrow t$  is  $C$ -provable. The theorem is a consequence of (12) and (13).
- (32)  $(t \vee u) \rightarrow v$  is  $C$ -provable if and only if  $t \rightarrow v$  is  $C$ -provable and  $u \rightarrow v$  is  $C$ -provable. The theorem is a consequence of (13), (21), and (31).
- (33) Suppose  $t \rightarrow v$  is  $C$ -provable and  $u \rightarrow w$  is  $C$ -provable. Then
  - (i)  $(t \vee u) \rightarrow (v \vee w)$  is  $C$ -provable, and
  - (ii)  $t \wedge u \rightarrow v \wedge w$  is  $C$ -provable.

The theorem is a consequence of (13), (32), and (29).

- (34)  $t \rightarrow u$  is  $C$ -provable if and only if  $u$  is  $(C \cup \{t\})$ -provable.

PROOF: Set  $D = C \cup \{t\}$ . If  $t \rightarrow u$  is  $C$ -provable, then  $u$  is  $D$ -provable by [2, (6)], (12).  $\square$

From now on  $D$  denotes a (SCI-axioms  $L$ )-extending subset of  $L$ .

Let us consider  $X$ ,  $L$ , and  $D$ . Let us note that every formula of  $L$  which is non  $D$ -provable is also non  $L$ -SCI-provable and there exists a formula of  $L$  which is  $D$ -provable.

Let us consider  $X$ ,  $L$ ,  $D$ ,  $t$ , and  $u$ . Now we state the propositions:

- (35) If  $t \equiv u$  is  $D$ -provable, then  $t \rightarrow u$  is  $D$ -provable. The theorem is a consequence of (12).
- (36) If  $t \equiv u$  is  $D$ -provable, then  $\neg t \equiv \neg u$  is  $D$ -provable. The theorem is a consequence of (12).
- (37)  $(t \equiv u) \rightarrow (u \equiv t)$  is  $D$ -provable. The theorem is a consequence of (13), (16), and (12).
- (38) If  $t \equiv u$  is  $D$ -provable, then  $u \equiv t$  is  $D$ -provable. The theorem is a consequence of (37) and (12).

Now we state the propositions:

- (39)  $(t \equiv u) \wedge (v \equiv u) \rightarrow (t \equiv v)$  is  $D$ -provable. The theorem is a consequence of (37), (13), (16), and (12).
- (40) If  $t$  is  $D$ -provable and  $t \Rightarrow u$  is  $D$ -provable, then  $u$  is  $D$ -provable. The theorem is a consequence of (35), (12), and (28).
- (41) Suppose  $t \equiv u$  is  $D$ -provable and  $v \equiv w$  is  $D$ -provable. Then

- (i)  $t \wedge v \equiv u \wedge w$  is  $D$ -provable, and
- (ii)  $t \wedge v \rightarrow u \wedge w$  is  $D$ -provable, and
- (iii) if  $t \wedge v$  is  $D$ -provable, then  $u \wedge w$  is  $D$ -provable, and
- (iv)  $(t \vee v) \equiv (u \vee w)$  is  $D$ -provable, and
- (v)  $(t \vee v) \rightarrow (u \vee w)$  is  $D$ -provable, and
- (vi) if  $t \vee v$  is  $D$ -provable, then  $u \vee w$  is  $D$ -provable, and
- (vii)  $(t \rightarrow v) \equiv (u \rightarrow w)$  is  $D$ -provable, and
- (viii)  $(t \rightarrow v) \rightarrow (u \rightarrow w)$  is  $D$ -provable, and
- (ix) if  $t \rightarrow v$  is  $D$ -provable, then  $u \rightarrow w$  is  $D$ -provable, and
- (x)  $(t \leftrightarrow v) \equiv (u \leftrightarrow w)$  is  $D$ -provable, and
- (xi)  $(t \leftrightarrow v) \rightarrow (u \leftrightarrow w)$  is  $D$ -provable, and
- (xii) if  $t \leftrightarrow v$  is  $D$ -provable, then  $u \leftrightarrow w$  is  $D$ -provable, and
- (xiii)  $(t \equiv v) \equiv (u \equiv w)$  is  $D$ -provable, and
- (xiv)  $(t \equiv v) \rightarrow (u \equiv w)$  is  $D$ -provable, and
- (xv) if  $t \equiv v$  is  $D$ -provable, then  $u \equiv w$  is  $D$ -provable, and
- (xvi)  $(t \Rightarrow v) \equiv (u \Rightarrow w)$  is  $D$ -provable, and
- (xvii)  $(t \Rightarrow v) \rightarrow (u \Rightarrow w)$  is  $D$ -provable, and
- (xviii) if  $t \Rightarrow v$  is  $D$ -provable, then  $u \Rightarrow w$  is  $D$ -provable.

The theorem is a consequence of (12) and (35).

Let us consider  $X$ ,  $L$ ,  $D$ ,  $t$ ,  $u$ , and  $v$ . Now we state the propositions:

- (42) (i)  $(t \equiv u) \rightarrow (t \wedge v \equiv u \wedge v)$  is  $D$ -provable, and
- (ii)  $(t \equiv u) \rightarrow (v \wedge t \equiv v \wedge u)$  is  $D$ -provable, and
  - (iii)  $(t \equiv u) \rightarrow ((t \vee v) \equiv (u \vee v))$  is  $D$ -provable, and
  - (iv)  $(t \equiv u) \rightarrow ((v \vee t) \equiv (v \vee u))$  is  $D$ -provable, and
  - (v)  $(t \equiv u) \rightarrow ((t \rightarrow v) \equiv (u \rightarrow v))$  is  $D$ -provable, and
  - (vi)  $(t \equiv u) \rightarrow ((v \rightarrow t) \equiv (v \rightarrow u))$  is  $D$ -provable, and
  - (vii)  $(t \equiv u) \rightarrow ((t \leftrightarrow v) \equiv (u \leftrightarrow v))$  is  $D$ -provable, and
  - (viii)  $(t \equiv u) \rightarrow ((v \leftrightarrow t) \equiv (v \leftrightarrow u))$  is  $D$ -provable, and
  - (ix)  $(t \equiv u) \rightarrow ((t \equiv v) \equiv (u \equiv v))$  is  $D$ -provable, and
  - (x)  $(t \equiv u) \rightarrow ((v \equiv t) \equiv (v \equiv u))$  is  $D$ -provable, and
  - (xi)  $(t \equiv u) \rightarrow ((t \Rightarrow v) \equiv (u \Rightarrow v))$  is  $D$ -provable, and
  - (xii)  $(t \equiv u) \rightarrow ((v \Rightarrow t) \equiv (v \Rightarrow u))$  is  $D$ -provable.

The theorem is a consequence of (13) and (29).

- (43) Suppose  $t \equiv u$  is  $D$ -provable. Then
- (i)  $t$  is  $D$ -provable iff  $u$  is  $D$ -provable, and
  - (ii)  $t \wedge v \equiv u \wedge v$  is  $D$ -provable, and
  - (iii)  $t \wedge v \rightarrow u \wedge v$  is  $D$ -provable, and
  - (iv) if  $t \wedge v$  is  $D$ -provable, then  $u \wedge v$  is  $D$ -provable, and
  - (v)  $v \wedge t \equiv v \wedge u$  is  $D$ -provable, and
  - (vi)  $v \wedge t \rightarrow v \wedge u$  is  $D$ -provable, and
  - (vii) if  $v \wedge t$  is  $D$ -provable, then  $v \wedge u$  is  $D$ -provable, and
  - (viii)  $(t \vee v) \equiv (u \vee v)$  is  $D$ -provable, and
  - (ix)  $(t \vee v) \rightarrow (u \vee v)$  is  $D$ -provable, and
  - (x) if  $t \vee v$  is  $D$ -provable, then  $u \vee v$  is  $D$ -provable, and
  - (xi)  $(v \vee t) \equiv (v \vee u)$  is  $D$ -provable, and
  - (xii)  $(v \vee t) \rightarrow (v \vee u)$  is  $D$ -provable, and
  - (xiii) if  $v \vee t$  is  $D$ -provable, then  $v \vee u$  is  $D$ -provable, and
  - (xiv)  $(t \rightarrow v) \equiv (u \rightarrow v)$  is  $D$ -provable, and
  - (xv)  $(t \rightarrow v) \rightarrow (u \rightarrow v)$  is  $D$ -provable, and
  - (xvi) if  $t \rightarrow v$  is  $D$ -provable, then  $u \rightarrow v$  is  $D$ -provable, and
  - (xvii)  $(v \rightarrow t) \equiv (v \rightarrow u)$  is  $D$ -provable, and
  - (xviii)  $(v \rightarrow t) \rightarrow (v \rightarrow u)$  is  $D$ -provable, and
  - (xix) if  $v \rightarrow t$  is  $D$ -provable, then  $v \rightarrow u$  is  $D$ -provable, and
  - (xx)  $(t \leftrightarrow v) \equiv (u \leftrightarrow v)$  is  $D$ -provable, and
  - (xxi)  $(t \leftrightarrow v) \rightarrow (u \leftrightarrow v)$  is  $D$ -provable, and
  - (xxii) if  $t \leftrightarrow v$  is  $D$ -provable, then  $u \leftrightarrow v$  is  $D$ -provable, and
  - (xxiii)  $(v \leftrightarrow t) \equiv (v \leftrightarrow u)$  is  $D$ -provable, and
  - (xxiv)  $(v \leftrightarrow t) \rightarrow (v \leftrightarrow u)$  is  $D$ -provable, and
  - (xxv) if  $v \leftrightarrow t$  is  $D$ -provable, then  $v \leftrightarrow u$  is  $D$ -provable, and
  - (xxvi)  $(t \equiv v) \equiv (u \equiv v)$  is  $D$ -provable, and
  - (xxvii)  $(t \equiv v) \rightarrow (u \equiv v)$  is  $D$ -provable, and
  - (xxviii) if  $t \equiv v$  is  $D$ -provable, then  $u \equiv v$  is  $D$ -provable, and
  - (xxix)  $(v \equiv t) \equiv (v \equiv u)$  is  $D$ -provable, and
  - (xxx)  $(v \equiv t) \rightarrow (v \equiv u)$  is  $D$ -provable, and
  - (xxxii) if  $v \equiv t$  is  $D$ -provable, then  $v \equiv u$  is  $D$ -provable, and

- (xxxii)  $(t \Rightarrow v) \equiv (u \Rightarrow v)$  is  $D$ -provable, and
- (xxxiii)  $(t \Rightarrow v) \rightarrow (u \Rightarrow v)$  is  $D$ -provable, and
- (xxxiv) if  $t \Rightarrow v$  is  $D$ -provable, then  $u \Rightarrow v$  is  $D$ -provable, and
- (xxxv)  $(v \Rightarrow t) \equiv (v \Rightarrow u)$  is  $D$ -provable, and
- (xxxvi)  $(v \Rightarrow t) \rightarrow (v \Rightarrow u)$  is  $D$ -provable, and
- (xxxvii) if  $v \Rightarrow t$  is  $D$ -provable, then  $v \Rightarrow u$  is  $D$ -provable.

The theorem is a consequence of (38), (35), (12), and (42).

#### 4. CONGRUENCES

Let us consider  $X$  and  $L$ .

A congruence of  $L$  is an equivalence relation of  $L$  defined by

- (Def. 47) for every  $t, u, v$ , and  $w$  such that  $\langle t, u \rangle, \langle v, w \rangle \in it$  holds  $\langle \neg t, \neg u \rangle, \langle t \wedge v, u \wedge w \rangle, \langle t \vee v, u \vee w \rangle, \langle t \rightarrow v, u \rightarrow w \rangle, \langle t \leftrightarrow v, u \leftrightarrow w \rangle, \langle t \equiv v, u \equiv w \rangle \in it$ .

In the sequel  $E$  denotes a congruence of  $L$ .

Let us consider  $X$  and  $L$ . One can verify that there exists a family of subsets of  $L$  which is non empty.

Let us consider  $E$ .

An equivalence class of  $E$  is an element of Classes  $E$ . Let us consider  $t$ . The functor  $E\text{-class } t$  yielding an equivalence class of  $E$  is defined by the term

- (Def. 48)  $[t]_E$ .

Now we state the proposition:

- (44)  $\langle t, u \rangle \in E$  if and only if  $E\text{-class } t = E\text{-class } u$ .

PROOF: If  $\langle t, u \rangle \in E$ , then  $E\text{-class } t = E\text{-class } u$  by [3, (18), (23)].  $\square$

From now on  $d, e$  denote equivalence classes of  $E$ .

Now we state the proposition:

- (45) There exists  $t$  such that  $d = E\text{-class } t$ .

Let us consider  $X, L, E$ , and  $d$ . The functor  $\neg d$  yielding an equivalence class of  $E$  is defined by

- (Def. 49) there exists  $t$  such that  $d = E\text{-class } t$  and  $it = E\text{-class } \neg t$ .

Let us consider  $e$ . The functors:  $d \wedge e, d \vee e, d \rightarrow e$ , and  $d \leftrightarrow e$  yielding equivalence classes of  $E$  are defined by conditions

- (Def. 50) there exists  $t$  and there exists  $u$  such that  $d = E\text{-class } t$  and  $e = E\text{-class } u$  and  $d \wedge e = E\text{-class } t \wedge u$ ,

- (Def. 51) there exists  $t$  and there exists  $u$  such that  $d = E\text{-class } t$  and  $e = E\text{-class } u$  and  $d \vee e = E\text{-class } (t \vee u)$ ,

(Def. 52) there exists  $t$  and there exists  $u$  such that  $d = E\text{-class } t$  and  $e = E\text{-class } u$  and  $d \rightarrow e = E\text{-class}(t \rightarrow u)$ ,

(Def. 53) there exists  $t$  and there exists  $u$  such that  $d = E\text{-class } t$  and  $e = E\text{-class } u$  and  $d \leftrightarrow e = E\text{-class}(t \leftrightarrow u)$ ,

respectively. Let us consider  $D$ . The functor  $\text{EqRel}(D)$  yielding a congruence of  $L$  is defined by

(Def. 54) for every  $t$  and  $u$ ,  $\langle t, u \rangle \in it$  iff  $t \equiv u$  is  $D$ -provable.

An equivalence class of  $D$  is an equivalence class of  $\text{EqRel}(D)$ . Let us consider  $t$ . The functor  $D\text{-class } t$  yielding an equivalence class of  $D$  is defined by the term

(Def. 55)  $\text{EqRel}(D)\text{-class } t$ .

Now we state the proposition:

(46)  $t \equiv u$  is  $D$ -provable if and only if  $D\text{-class } t = D\text{-class } u$ . The theorem is a consequence of (44).

In the sequel  $x, y, z$  denote equivalence classes of  $D$ .

Now we state the proposition:

(47) There exists  $t$  such that  $x = D\text{-class } t$ . The theorem is a consequence of (45).

Let us consider  $X, L, D$ , and  $x$ . We say that  $x$  is  $D$ -provable if and only if

(Def. 56) there exists  $t$  such that  $x = D\text{-class } t$  and  $t$  is  $D$ -provable.

Now we state the proposition:

(48)  $y = \neg x$  if and only if there exists  $t$  such that  $x = D\text{-class } t$  and  $y = D\text{-class } \neg t$ .

Let us consider  $X, L$ , and  $D$ . Let  $t$  be a  $D$ -provable formula of  $L$ . Let us note that  $D\text{-class } t$  is  $D$ -provable and there exists an equivalence class of  $D$  which is  $D$ -provable.

Now we state the proposition:

(49) If  $D\text{-class } t$  is  $D$ -provable, then  $t$  is  $D$ -provable. The theorem is a consequence of (46), (35), and (12).

Let us consider  $X, L$ , and  $D$ . Let  $x$  be a  $D$ -provable equivalence class of  $D$ . One can verify that every element of  $x$  is  $D$ -provable.

Let us consider  $x$  and  $y$ . Now we state the propositions:

(50)  $x \wedge y$  is  $D$ -provable if and only if  $x$  is  $D$ -provable and  $y$  is  $D$ -provable. The theorem is a consequence of (47), (49), and (28).

(51)  $x \equiv y$  is  $D$ -provable if and only if  $x = y$ . The theorem is a consequence of (47), (46), and (49).

Now we state the propositions:

- (52) (i)  $D\text{-class } \neg t = \neg(D\text{-class } t)$ , and  
(ii)  $D\text{-class } t \wedge u = (D\text{-class } t) \wedge (D\text{-class } u)$ , and  
(iii)  $D\text{-class}(t \vee u) = (D\text{-class } t) \vee (D\text{-class } u)$ , and  
(iv)  $D\text{-class}(t \rightarrow u) = (D\text{-class } t) \rightarrow (D\text{-class } u)$ , and  
(v)  $D\text{-class}(t \leftrightarrow u) = (D\text{-class } t) \leftrightarrow (D\text{-class } u)$ , and  
(vi)  $D\text{-class}(t \equiv u) = (D\text{-class } t) \equiv (D\text{-class } u)$ .
- (53) If  $x$  is  $D$ -provable, then  $x \vee y$  is  $D$ -provable and  $y \vee x$  is  $D$ -provable. The theorem is a consequence of (47) and (49).

Let us consider  $X$ ,  $L$ ,  $D$ ,  $t$ , and  $u$ . Now we state the propositions:

- (54)  $t \leftrightarrow u$  is  $D$ -provable if and only if  $t \rightarrow u$  is  $D$ -provable and  $u \rightarrow t$  is  $D$ -provable. The theorem is a consequence of (12) and (28).
- (55) If  $t \leftrightarrow u$  is  $D$ -provable, then  $u \leftrightarrow t$  is  $D$ -provable. The theorem is a consequence of (54).

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